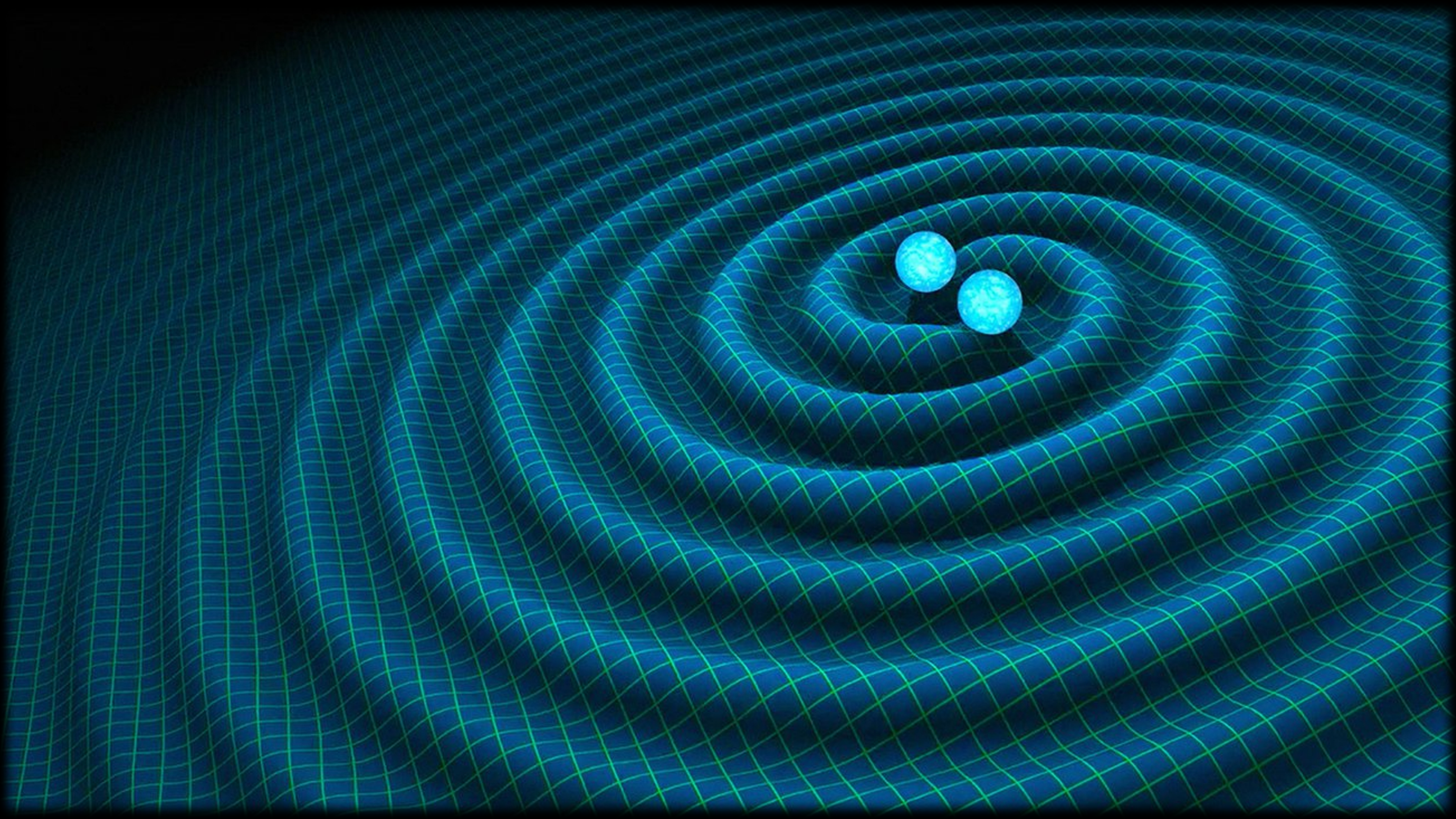
# General Relativity



Fabio Cufino

# 1 - Introduction

General Relativity is not a Quantum Theory

## Useful Notations

1. **Event**: A physica occurence at some point in space and time

2. **Observer**: A reference Frame
   N.P.: we can swith observer through coordinate transformations.

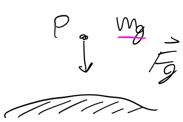
3. **Lael time**: Each observer is equipped with an idea standard clock and he can assign a temporal order to events
   → He can tell if an event cames early than others

4. **World line**: The path that an object follows in space time

---

## The Equivalence Principle

### 1. Galilean Version




1. $\vec{F_g} = -m_g \vec{\nabla} \varphi$
   ↳ gravitational mass

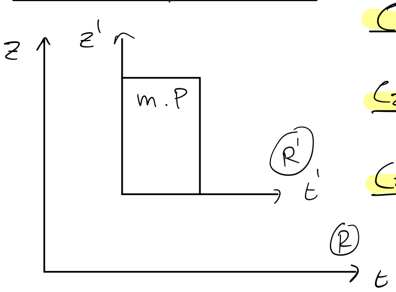
2. $\vec{F} = m_i \vec{a}$ ~ Tells how much an object resist to SOMETHING
   ↳ inertial mass

The 2 equation are FUNDAMENTALLY DIFFERENT OBJECTS

Empirically $m_i = m_g$ ⤳ Some experiments did that

CONSEQUENCE ⇒ Universally Free-Fall (all the objects Fall at the same speed)

### 2. Einstein Version




**Case 0**: Elevator in outher space Free of all Forces
→ P Floats

**Case 1**: Elevator in outher space uniformally accellerated
→ P moves towards bottom

**Case 2**: Elevator get closer to Earth
→ P moves towards bottom

Einstein Principle
You cannot distinguish between this 2 objects

⟶ Gravity does not depends on the mass

## Solving case 1:

$z(t) = \frac{1}{2} g t^2 \Rightarrow \ddot{z}(t) = g$ and $z' = z - \frac{1}{2} g t^2$

$t' = t$ (ignore relativistic effects)

In R: $F = m\ddot{z}$

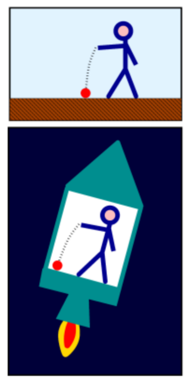
In R': $F' = m\ddot{z}' = m\ddot{z} - mg \Rightarrow F' = F - mg$ ⟶ The acceleration of R' respect R looks like gravitational Force in R'

**Case 3:** Elevator in the Earth Field and cut the rope

⟶ P Floats is equal to 0

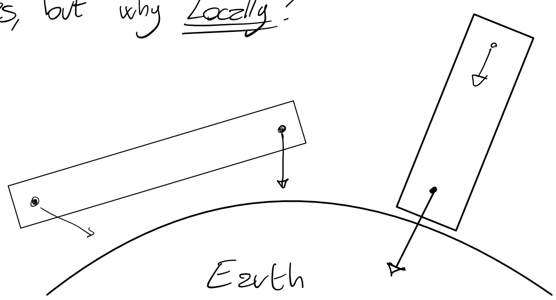
So this explain Einstein Principle:

"No physics experiment can distinguish locally between a uniform grav. Field acting on the local inertial observer and a uniform acceleration whith respect to that local inertial observer. The local inertial observer is the one for wich SR holds"



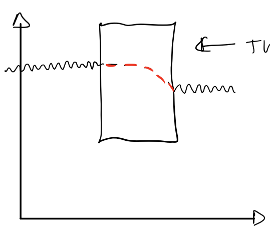
A falling object drops exactly the same on a planet or in an accelerating frame of reference

Yes, but why Locally?




The gravitational Field change in space, everything must be done locally

## Light Discussion

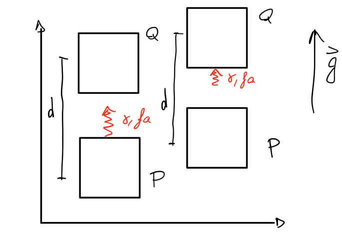


⟵ The elevator move! In an accelerated reference frame Light bend on the action by a gravitational Field

- light exit a little bit lower

⟶ Light Bends

## Gravitational Redshift




- Photons times of flight $\Delta t = \frac{d}{c}$
- Velocity increase for P, Q

$$\Delta v = g\Delta t \implies \Delta v = \frac{gd}{c}$$

- Doppler shift:

$$\frac{f_a}{f_p} = 1 - \frac{\Delta v}{c} = 1 - \frac{gd}{c^2}$$

Because of the equivalence principle

# 2 - Special Relativity

## 1. Events: (single point)

$(x^0, x^1, x^2, x^3)$ where $x^0 = ct$

$\{x^i\}_{i=1,2,3}$ = position space $\vec{x}$

This is how we label events on Minkowsky Space-Time

## 2. Distance:

$$ds^2 = (x^0)^2 - \sum_{\mu} (x^{\mu})^2 \qquad \text{LINE ELEMENT}$$

Alternative conventions: $ds^2 = -(x^0)^2 + \sum_{\mu} (x^{\mu})^2$

## 3. Matrix Tensor

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- it's invariant

Using that the "LINE ELEMENT" is: $ds^2 = \sum_{\mu,\nu} \eta_{\mu\nu}\, dx^{\mu} dx^{\nu}$

## 4. Einstein Summation Convention

Repeated indices are summed $\Rightarrow$ $ds^2 = \eta_{\mu\nu}\, dx^{\mu} dx^{\nu}$

## 5. Lorentz Transformation

Transformation
- LINEAR TRANSFORMATIONS
- Leave $ds^2$ INVARIANT $\rightarrow$ $x^{\mu} \rightarrow \boxed{x'^{\mu} = L^{\mu}{}_{\nu}\, x^{\nu}}$ → Lorentz Transformation

$\rightsquigarrow x'^{\mu} = L^{\mu}{}_{1} x^1 + L^{\mu}{}_{2} x^2 + L^{\mu}{}_{3} x^3 + \dots$

such that $ds'^2 = \eta_{\mu\nu}\, dx'^{\mu} dx'^{\nu} = \eta_{\mu\nu}\, dx^{\mu} dx^{\nu} = ds^2$

Consequence: $\boxed{\eta_{\mu\nu} L^{\mu}{}_{\alpha} L^{\nu}{}_{\beta} = \eta_{\alpha\beta}}$ → it's a consequence because we already know that it's invariant!

## 6. Lorentz + Translation : Poincaré

$$x^\mu \to x'^\mu = L^\mu{}_\nu x^\nu + \xi^\mu$$

→ Inertial reference frame are related through Poincaré Transformation

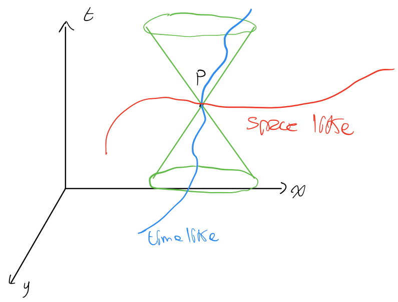
Remark

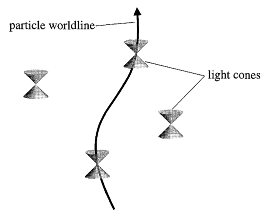
Loren/Poinc Transformations form a Group

## 7. Distance of 2 events

$$\Delta x^2 = \eta_{\mu\nu}(x_P^\mu - x_Q^\mu)(x_P^\nu - x_Q^\nu)$$

- if $\Delta x^2 > 0 \to P,Q$ "Time like separated"
- if $\Delta x = 0 \to P,Q$ "Light like separated"
- if $\Delta x^2 < 0 \to P,Q$ "Space like separated"




## 8. Parametrize Curve:

$\lambda \to x^\mu(\lambda)$, $\lambda \in I \subseteq \mathbb{R}$

Tangent Vector: $V^\mu = \dfrac{dx^\mu(\lambda)}{d\lambda}$ (More correct $V^\mu(\lambda_0) = \dfrac{dx^\mu(\lambda)}{d\lambda}\Big|_{\lambda=\lambda_0}$)

N.B.: We can reparametrize curves writing $\Phi = \psi(\lambda)$ $\Phi \in J \subseteq \mathbb{R}$

$\Rightarrow c' = c \circ \psi^{-1}$ : $c(J) = c(I)$

Basically THE CURVE remain THE SAME

Common choise: $\lambda : \tau = \sqrt{\eta_{\mu\nu} x^\mu x^\nu}$ PROPER TIME

but why it's good?
- it's time measured by an observer in their own frame
- it's LORENTZ INVARIANT → other measures different time

Tangent vector of τ-parametrised curve:

$$\frac{d}{d\tau'}\left(x'^{\mu}(\tau')\right)\Big|_{\tau'=\tau} = \frac{d}{d\tau}\left(x'^{\mu}(\tau)\right) = \frac{\partial x'^{\mu}}{\partial x^{\nu}}\frac{dx^{\nu}(\tau)}{d\tau} = L^{\mu}{}_{\nu}\frac{dx^{\nu}(\tau)}{d\tau}$$

## 9. Lorentz Vector:

$$V'^{\mu} = L^{\mu}{}_{\nu} V^{\nu}$$

## 10. Scalar Product

Two vectors $V, W$ components $V^{\mu}, W^{\mu}$

$$\left.\begin{array}{l} \eta_{\mu\nu} V^{\mu} W^{\nu} \\ \text{Norm: } \eta_{\mu\nu} V^{\mu} V^{\nu} \end{array}\right\} \text{Lorentz Invariant}$$

Notation

$$* \quad V^{\mu} V_{\nu} = \eta_{\mu\nu} V^{\mu} V_{\nu} \quad *$$

## 11. Covector Transforms

(conjugate transpose of $L$, $\Lambda \equiv (L^{T})^{-1}$)

$$L^{T}\eta L = \eta \implies \Lambda = \eta L \eta^{-1}$$

- In index notation: $V'_{\mu} = \Lambda^{\nu}{}_{\mu} V_{\mu}$ (2)

$$\Lambda^{\nu}_{\mu} = \eta_{\mu\alpha} L^{\alpha}{}_{\beta}\, \eta^{\beta\nu}$$

- Covector: lower indices

Covector examples:

1) $u_{\alpha} = \eta_{\alpha\beta} V^{\beta}$ with $V^{\beta}$ Lorentz vector

check if it's a covector using (2): 

$$u'_{\mu} = \left(\eta_{\alpha\beta} V^{\beta}\right)' = \underbrace{\eta_{\alpha\beta}}_{\text{invariant}} V'^{\beta} = \eta_{\alpha\beta} L^{\beta}{}_{\nu} V^{\nu}$$

But $u_{\alpha} = \eta_{\alpha\beta} V^{\beta} \implies V^{\beta} = \eta^{\beta\alpha} u_{\alpha}$

$$\rightarrow \quad u'_{\alpha} = \underbrace{\eta_{\alpha\beta} L^{\beta}{}_{\nu}\, \eta^{\nu\gamma}}_{\Lambda^{\gamma}_{\alpha}} u_{\gamma} = \Lambda^{\gamma}_{\alpha} u_{\gamma}$$

2) Partial derivative of a scalar function

$$W_\mu = \frac{\partial \varphi}{\partial x^\mu}$$ is a covector ( $\varphi(x^\mu)$ scalar function)

ex: prove it

! Vector exist regardless of basis

$$V = V^\mu e_{(\mu)} \longrightarrow V = V^1 e_{(1)} + V^2 e_{(2)} + V^3 e_{(3)} + V^4 e_{(4)}$$

basis

basis dependent

basis independent components

## 12. Tensors

Lorentz $(p,q)$ tensor object that transform as a product of $p$ vectors and $q$ covector take $T^{\mu_1,\dots,\mu_p}_{\nu_1,\dots,\nu_q}$

$\Lambda$ Tran:

$$T'^{\mu_1,\dots,\mu_p}_{\nu_1,\dots,\nu_q} = \Lambda^{\mu_1}_{\alpha_1}\cdots\Lambda^{\mu_p}_{\alpha_p}\Lambda^{\beta_1}_{\nu_1}\cdots\Lambda^{\beta_q}_{\nu_q}\, T^{\mu_1,\dots,\mu_p}_{\nu_1,\dots,\nu_q}$$

Examples:

1) $\eta_{\mu\nu}$ : $(0,2)$ tensor
2) $\eta^{\mu\nu}$ : $(2,0)$ tensor
3) $\delta^\mu_\nu$ : $(1,1)$ tensor

Using this we can associate COVECTOR $\leftrightarrow$ VECTOR

$$V^\mu \to V_\mu = \eta_{\mu\nu}V^\nu$$

$$V_\mu \to V^\mu = \eta^{\mu\nu}V_\nu$$

Additional property: $\eta^{\mu\nu}\eta_{\nu\rho} = \delta^\mu_\rho$ (Kronek. delt)

4) Sums of $(p,q)$ tensors: $(q,p)$ tensor

5) Given $V^\mu, W^\nu, U_\rho \to V^\mu W^\nu U_\rho$ is a $(2,1)$ tensor

# WORLD LINE OF MASSIVE PARTICLE

Assuming that $x^\mu = x^\mu(\tau)$

- **Velocity**: $V^\mu \equiv \frac{dx^\mu}{d\tau}$ Lorentz Vector, normalized as $V^\mu V_\mu = c^2$

- **Acceleration**: $a^\mu \equiv \frac{dV^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2}$ For free particle $a^\mu = 0$

  For an accelerating observer $a^\mu V_\mu = 0$

- **Action**:

  For a free massive particle

$$S = -mc^2 \int d\tau = -mc^2 \int \sqrt{\eta_{\mu\nu}\, dx^\mu dx^\nu}$$

  ( Reminder: useful due to least action principle)

  Whith respect to an arbitrary parameter $\lambda$:

$$S = \int L_\lambda d\lambda \qquad \text{where} \qquad L_\lambda = -mc \left[ \eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right]^{1/2}$$

  if we chose $\lambda = t$, then

$$L_t = -mc^2 \sqrt{1 - \frac{\vec{v}^2}{c^2}} \quad , \quad \vec{v} \equiv \frac{d\vec{x}}{dt}$$

- **Momentum**:

$$P_\mu = \frac{\partial L_\lambda}{\partial (dx^\mu/d\lambda)} = m\eta_{\mu\nu} x^\nu \Rightarrow P_\mu = m\left(\frac{dx^\mu}{d\tau}\right)$$

  Components $P^0 = \frac{E}{c}$ , $P^k = m\gamma V^k$ $(k = 1,2,3)$ (latin letters $1 \to 3$) (letters $0 \to 3$)

  where $\gamma = \left(1 - \frac{\vec{v}^2}{c^2}\right)^{-1/2}$ and $E = m\gamma c^2$

$$\Rightarrow P^\mu P_\mu = m^2c^2 \Leftrightarrow E^2 = m^2c^4 + \vec{p}^2c^2$$

# In General Relativity Space Time is a 4D MANIFOLD

## Definition: Manifold $M$

$M$ is a differentiable manifold $M$ of dimension $n \in \mathbb{N}^+$

i) $M$ is a Hausdorff topological space (define neighborhoods which are distinct)

ii) $\exists$ a Family of open sets $\{U_i\}$ $i \in I$ (set of index) which covers $M$: $\bigcup_{i \in I} (U_i) = M$
$\rightarrow$ There are spaces that covers all the manifold

iii) Each set $U_i$ can be mapped onto open subset of $\mathbb{R}^n$ through a continuous INVERTIBLE map $\phi_i : U_i \rightarrow U_i' \subset \mathbb{R}^n$

iv) Given 2 sets $U_i, U_j \subset M$ with $U_i \cap U_j \neq \emptyset$ the map $\psi_{i,j} \equiv \phi_j \circ \phi_i^{-1}$ from $\phi_i(U_i \cap U_j) \rightarrow \phi_j(U_i \cap U_j)$ is infinitely differentiable

---

## Terminology

**Chart**: A pair $(U_i, \phi_i)$

**Atlas**: The set $\{U_i, \phi_i\}$

**Coordinate Neighbourhood**: $U_i$

**Coordinate Function**: $\phi_i$

**Coordinates**: In $\mathbb{R}^n$ $\phi_i$ is represented by a set of Functions $X^1(P), .., X^n(P)$ $\forall P \in U_i$

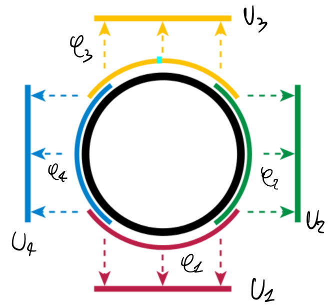



Figure 1: The four charts each map part of the circle to an open interval, and together cover the whole circle.

( , , , ) = Atlas

$\longrightarrow$ This definition is telling that if I consider a differential MANIFOLD what i can do:

- I can range locally $\forall P \in M$ to $\mathbb{R}^n$, do NORMAL TENSOR CALCULUS, map back to the manifold, move around by going into different neighborhood subspaces, define coordinates by again mapping into $\mathbb{R}^n$, do NORMAL CALCULUS and so on...

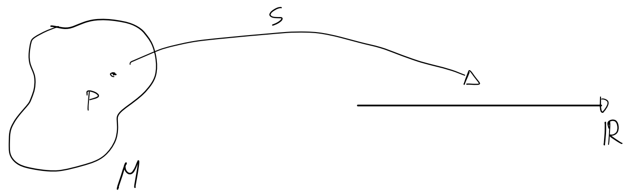
---

$M$ LOCALLY LOOKS LIKE $\mathbb{R}^n$

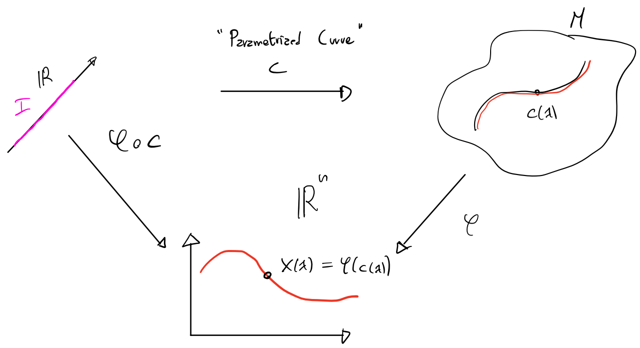
# Definitions

**Scalar:** A scalar in Space-Time is a smooth map $S: M \to \mathbb{R}$.
To each point $p \in M$ it associate a number $S(p) \in \mathbb{R}$

$\rightsquigarrow$ Every observer agree on $P$ (in all the systems):
Under a coordinate transformation $x^{\mu} \to x'^{\mu}$ we have $S'(x'^{\mu}) = S(x^{\mu})$




**Curve:** A parametrised curve on a $M$ is a smooth map $c: I \subseteq \mathbb{R} \to M$ which to any real $\lambda \in I$ (parameter) associates a point in $M$




**Co-vector:** A (contravariant, tangent) vector

i) $$\boxed{V'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu}}$$ $\left( \sim x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \right)$

Example:

$$dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} dx^{\nu}$$

The set of all t-vectors is called Vector space $T_P$ (Tangent Space)

Analogy $\to$ $V = V^{\mu} \partial_{\mu} = \partial_{\mu} V^{\mu}$

$\hookrightarrow$ basis vectors

$\to$ Contravariant

ii) A covector is an object whit components $U_i$ trasforming as:

$$\boxed{U'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} U_\nu}$$

Covectors form a CO-TANGENT space

**Tensors:** A $(p,q)$ tensor is an object with components

$$T'^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q} = \frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\mu_p}}{\partial x^{\alpha_p}} \frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\beta_q}}{\partial x'^{\nu_q}} T^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q} \qquad (p,q) \text{ Tensor}$$

Example: Kronecker $\delta^\mu{}_\nu = \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu \end{cases}$

$$\delta^\mu{}_\nu = \frac{\partial x^\mu}{\partial x^\nu} = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x'^\alpha}{\partial x^\nu} = \frac{\partial x^\mu}{\partial x'^\alpha} \underbrace{\frac{\partial x'^\alpha}{\partial x'^\beta}}_{\delta'^\alpha{}_\beta} \frac{\partial x'^\beta}{\partial x^\nu} = \underbrace{\frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x'^\beta}{\partial x^\nu}}\, \delta'^\alpha{}_\beta$$

$(1,1)$ Tensor $\leftarrow$ $\delta'^\alpha{}_\beta = \dfrac{\partial x^\nu}{\partial x'^\beta} \dfrac{\partial x'^\alpha}{\partial x^\mu} \delta^\mu{}_\nu$

By inverting both $\partial$ from the derivatives

Example: Given 2 vectors $V^\mu$, $W^\nu$, $(V^\mu W^\nu)$ is a $(2,0)$ tensor

$$V'^\mu W'^\nu = \frac{\partial x'^\mu}{\partial x^\beta} V^\beta \frac{\partial x'^\nu}{\partial x^\alpha} W^\alpha = \frac{\partial x'^\mu}{\partial x^\beta} \frac{\partial x'^\nu}{\partial x^\alpha} V^\beta W^\alpha \qquad (2,0) \text{ Tensor}$$

---

## Operations with Tensors

i) Multiplication by $a \in \mathbb{R}$

if $T$ is $(p,q)$-T, $a\,T$ is $(p,q)$-T $\forall a \in \mathbb{R}$

ii) Addition of $(p,q)$ tensors $\rightarrow$ $(p,q)$ tensors

iii) Tensor Product: $V^\mu$, $W_\nu$ : $V \otimes W = V^\mu W_\nu \equiv T^\mu{}_\nu$

iv) Contraction of indices: $T^{\mu_1 \dots \mu_k \dots \mu_p}_{\nu_1 \dots \mu_k \dots \nu_q} \rightarrow$ Sum Conv $\Rightarrow = T^{\mu_1 \dots 0 \dots \mu_p}_{\nu_1 \dots 0 \dots \nu_q} + \dots + T^{\mu_1 \dots 0 \dots \mu_p}_{\nu_1 \dots 0 \dots \nu_q} = M^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q}$ elim. That Index $(M \neq T)$

v) Covariant Differentiation: $S(x)$, $V(x)$, $T(x)$

## Definition: Metric Tensor $g_{\mu\nu}(x)$

A $(0,2)$ tensor field with two basic properties

~ it describes the geometry of the manifolds

~ it defines the scalar product

1) $g_{\mu\nu}(x) = g_{\nu\mu}(x)$ (symmetric)

2) $\det [g_{\mu\nu}(x)] \neq 0$ (non-degenerate)

\+ it preserves the "line element" under $x^\mu \to x'^\mu$:

$$ds^2(x) = g_{\mu\nu}\, dx^\mu dx^\nu$$

$$ds'^2(x') = g_{\mu\nu}(x')\, dx'^\mu dx'^\nu = ds$$

**Exercise: Show that $g_{\mu\nu}$ is $(0,2)$ tensor.**

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu \qquad ①$$

$$ds'^2 = g'_{\mu\nu}(x)\, dx'^\mu dx'^\nu \qquad ④$$

but $dx'^\mu = \dfrac{\partial x'^\mu}{\partial x^\alpha} dx^\alpha \quad \Rightarrow \quad dx^\mu = \dfrac{\partial x^\mu}{\partial x'^\alpha} dx'^\alpha \qquad ②$

and $dx'^\nu = \dfrac{\partial x'^\nu}{\partial x^\beta} dx^\beta \quad \Rightarrow \quad dx^\nu = \dfrac{\partial x^\nu}{\partial x'^\beta} dx'^\beta \qquad ③$

1, 2, 3 $\Rightarrow$ $ds^2 = g_{\mu\nu}(x) \dfrac{\partial x^\mu}{\partial x'^\alpha} \dfrac{\partial x^\nu}{\partial x'^\beta} dx'^\alpha dx'^\beta$

$$= g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} dx'^\mu dx'^\nu$$

From ④ we have $ds'^2 = ds$ so

$$g'_{\mu\nu}(x) = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \qquad (0,2) \text{ tensor}$$

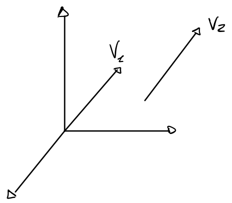
## Definition: Inverse Metric

$$g_{\mu\nu}\, g^{\nu\alpha} = \delta_\mu{}^\alpha$$

Raising / Lowering indices:

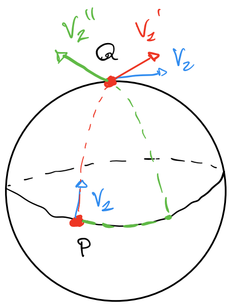
1) From a vector $V^\mu$ we can construct a covector $V_\mu \equiv g_{\mu\nu} V^\nu$
2) Given the inverse metric, a covector $V_\mu$ we can construct a vector $V^\mu = g^{\mu\nu} V_\nu$

# MANIFOLD




If you want to compare 2 vector in $\mathbb{R}^3$, you take $V_2$ and translate it near $V_1$

We want to do the same in MANIFOLD




Now on a surface i want to do the same, so i connect the 2 points and translate along that path, $V_1$ become $V_1'$

Another path could be the green, but the 2 translation gives 2 diff vectors

→ The Final vector DEPENDS on the PATH

---

Considering a vector A with components $A^\mu$, the $\partial_\nu A^\mu$ is Tensor?

So

$$\left.\begin{aligned} \partial'_\mu &= \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial}{\partial x^\alpha} \\ A'^\mu &= \frac{\partial x'^\nu}{\partial x^\beta} A^\beta \end{aligned}\right\} \quad (\partial_\mu A^\nu)' = \partial'_\mu A'^\mu = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial}{\partial x^\alpha}\left(\frac{\partial x'^\nu}{\partial x^\beta} A^\beta\right)$$

$$= \underbrace{\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x'^\nu}{\partial x^\beta}\partial_\alpha A^\beta}_{\text{Tensor - like}} + \underbrace{\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial^2 x'^\nu}{\partial x^\alpha \partial x^\beta} A^\beta}_{\text{Not tensor}}$$

⟹ Partial Derivatives is not a Tensor

---

**Theorem:** For every metric $g_{\mu\nu}$ it's always possible to find coordinates ("Riemann norm.. coordinates) such that at one point $P = x_0 \equiv 0 \equiv x_0'$ :

1) $g'_{\mu\nu}(x_0^\circ) = \eta_{\mu\nu}$

2) $\partial_{\rho'} g'_{\mu\nu}\big|_{x_0} = 0$

3) $\partial_{\rho'}\partial_{\sigma'} g'_{\mu\nu}(x')\big|_{x_0'} \neq 0$ unless the space is flat

**Definition:** COVARIANT DERIVATIVE

$$\boxed{D_\mu A^\nu \equiv \partial_\mu A^\nu + \Gamma^\nu_{\mu\lambda} A^\lambda}$$

→ Demand to be a tensor

$\Gamma^\nu_{\mu\lambda}A^\lambda$ → How much the coordinate change

$\partial_\mu A^\nu$ → "How much the vector changes"

$\Gamma$ : "christoffer symbol"

"Afine connection coefficients"

• Christoffer symbol is a tensor?

Answer: $\Gamma^\nu_{\mu\lambda}$ → Not Tensor. Proof why:

– Useful relations:

1. $D_\mu V^\nu \equiv \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$

2. $V'^\mu = \dfrac{\partial x'^\mu}{\partial x^\nu} V^\nu \iff V^\nu = \dfrac{\partial x^\nu}{\partial x'^\mu} V'^\mu$

3. $V'_\mu = \dfrac{\partial x^\alpha}{\partial x'^\mu} V_\alpha \iff V_\alpha = \dfrac{\partial x'^\mu}{\partial x^\alpha} V'_\mu$

4. $T'^\mu{}_\nu = \dfrac{\partial x'^\mu}{\partial x^\alpha} \dfrac{\partial x^\beta}{\partial x'^\nu} T^\alpha{}_\beta \implies T^\alpha{}_\beta = \dfrac{\partial x^\alpha}{\partial x'^\mu} \dfrac{\partial x'^\nu}{\partial x^\beta} T'^\mu{}_\nu$

$$D_\mu V^\nu \overset{④}{=} \frac{\partial x^\nu}{\partial x'^\alpha}\frac{\partial x'^\beta}{\partial x^\mu}(D_\beta V^\alpha)'$$

$$\overset{①}{=} \frac{\partial x^\nu}{\partial x'^\alpha}\frac{\partial x'^\beta}{\partial x^\mu}\left[\partial_\beta V^\alpha + \Gamma^\alpha_{\beta\rho} V^\rho\right]$$

let's see how [ ] transforms

• $\left(\partial_\beta V^\alpha + \Gamma^\alpha_{\beta\rho}V^\rho\right)' \overset{⑤}{=} \underbrace{\dfrac{\partial x^k}{\partial x'^\beta}\dfrac{\partial x'^\alpha}{\partial x^\lambda}\partial_k V^\lambda + \dfrac{\partial x^k}{\partial x'^\beta}\dfrac{\partial^2 x}{\partial x^k \partial x^\lambda}V^\lambda}_{\text{From last page}} + \Gamma'^\alpha_{\beta\rho}\underbrace{\dfrac{\partial x'^\rho}{\partial x^\lambda}V^\lambda}_{\text{From 3. For } V^\lambda}$

Then:

$$D_\mu V^\nu = \frac{\partial x^\nu}{\partial x'^\alpha}\frac{\partial x'^\beta}{\partial x^\mu}\left[\frac{\partial x^k}{\partial x'^\beta}\frac{\partial x'^\alpha}{\partial x^\lambda}\partial_k V^\lambda + \frac{\partial x^k}{\partial x'^\beta}\frac{\partial^2 x'^\alpha}{\partial x^k \partial x^\lambda}V^\lambda + \Gamma'^\alpha_{\beta\rho}\frac{\partial x'^\rho}{\partial x^\lambda}V^\lambda\right]$$

but $\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x'^\alpha}{\partial x^\nu} = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu{}_\nu$

$$\Rightarrow D_\mu V^\nu = \delta^\nu{}_\lambda \delta^\kappa{}_\mu \partial_\kappa V^\lambda + \delta^\kappa{}_\mu \frac{\partial x^\nu}{\partial x'^\alpha}\frac{\partial^2 x^\nu}{\partial x^\kappa \partial x^\lambda} V^\lambda + \frac{\partial x^\nu}{\partial x'^\alpha}\frac{\partial x'^\beta}{\partial x^\mu}\frac{\partial x'^\rho}{\partial x^\lambda}\Gamma'^\alpha{}_{\beta\rho} V^\lambda$$

Using $\underline{\underline{\delta^\nu{}_\lambda V^\lambda = V^\nu}}$

$$\Rightarrow D_\mu V^\nu = \partial_\mu V^\nu + \frac{\partial x^\nu}{\partial x'^\alpha}\frac{\partial^2 x'^\alpha}{\partial x^\mu \partial x^\lambda} \underline{\underline{V^\lambda}} + \frac{\partial x^\nu}{\partial x'^\alpha}\frac{\partial x'^\beta}{\partial x^\mu}\frac{\partial x'^\rho}{\partial x^\lambda}\Gamma'^\alpha{}_{\beta\rho} \underline{\underline{V^\lambda}} \qquad (A)$$

And $D_\mu V^\nu \equiv \partial_\mu V^\nu + \Gamma^\nu{}_{\mu\lambda}\underline{\underline{V^\lambda}} \qquad (1)$

(A) and (1) must hold $\underline{\underline{\forall \lambda}}$ so:

$$\Gamma^\nu{}_{\mu\lambda} = \frac{\partial x^\nu}{\partial x'^\alpha}\frac{\partial^2 x'^\alpha}{\partial x^\mu \partial x^\lambda} + \frac{\partial x^\nu}{\partial x'^\alpha}\frac{\partial x'^\beta}{\partial x^\mu}\frac{\partial x'^\rho}{\partial x^\lambda}\Gamma'^\alpha{}_{\beta\rho} \Rightarrow \text{rearrange}$$

$$\Gamma'^\alpha{}_{\beta\rho} = \underbrace{\frac{\partial x'^\alpha}{\partial x^\nu}\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\lambda}{\partial x'^\beta}\Gamma^\nu{}_{\mu\lambda}}_{\text{Tensor-like}} - \underbrace{\frac{\partial x^\mu}{\partial x'^\beta}\frac{\partial x^\lambda}{\partial x'^\rho}\frac{\partial^2 x'^\alpha}{\partial x^\mu \partial x^\lambda}}_{\text{NOT Tensor like}}$$

Consider $\Gamma^\nu{}_{\mu\lambda}$, $\bar{\Gamma}^\nu{}_{\mu\lambda} = \Gamma^\nu{}_{\lambda\mu}$

$T^\nu{}_{\mu\lambda} \equiv \Gamma^\nu{}_{\mu\lambda} - \bar{\Gamma}^\nu{}_{\mu\lambda} \rightarrow$ "Torsion" Exercise: Prove $T$ is a $(1,2)$ tensor

---

$$T'^\nu{}_{\mu\lambda} = \Gamma'^\nu{}_{\mu\lambda} - \Gamma'^\nu{}_{\lambda\mu} = \frac{\partial x'^\alpha}{\partial x^\nu}\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\lambda}{\partial x'^\beta}\Gamma^\nu{}_{\mu\lambda} - \cancel{\frac{\partial x^\mu}{\partial x'^\beta}\frac{\partial x^\lambda}{\partial x'^\rho}\frac{\partial^2 x'^\alpha}{\partial x^\mu \partial x^\lambda}} -$$

Dummy index

$$- \left( \frac{\partial x'^\alpha}{\partial x^\nu}\frac{\partial x^\lambda}{\partial x'^\rho}\frac{\partial x^\mu}{\partial x'^\beta}\Gamma^\nu{}_{\lambda\mu} - \cancel{\frac{\partial x^\lambda}{\partial x'^\beta}\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial^2 x'^\alpha}{\partial x^\lambda \partial x^\mu}} \right)$$

$$= \frac{\partial x'^\alpha}{\partial x^\nu}\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\lambda}{\partial x'^\beta}\left(\Gamma^\nu{}_{\mu\lambda} - \Gamma^\nu{}_{\lambda\mu}\right) \qquad (1,2) \text{ Tensor}$$

**Definition:** PARTIAL DERIVATIVE ALONG ANOTHER VECTOR

$$D_A V^\mu = A^\nu D_\nu V^\mu$$

**Definition:** COVARIANT DERIVATIVE OF $(p,q)$ TENSORS

1) $D_\mu$ is a linear operator that takes $(p,q) \to (p,q+1)$ tensor

2) If $S$ is a scalar $D_\mu S = \partial_\mu S$

3) $D_\mu \left( A^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q} B^{\rho_1 \dots \rho_k}_{\lambda_1 \dots \lambda_s} \right) = \left( D_\mu A^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q} \right) B^{\rho_1 \dots \rho_k}_{\lambda_1 \dots \lambda_s} + A^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q} \left( D_\mu B^{\rho_1 \dots \rho_k}_{\lambda_1 \dots \lambda_s} \right)$

Note That: $D_\mu U_\nu = \partial_\mu U_\nu - \Gamma^\lambda_{\mu\nu} U_\lambda$ $\qquad U_\nu = g_{\nu\mu} U^\mu$

Dim: Scalar $A_\nu B^\nu$, but $D_\mu (A_\nu B^\rho) = \partial_\mu (A_\nu B^\rho)$

$$\stackrel{!}{=} (D_\mu A_\nu) B^\rho + A_\nu (D_\mu B^\rho)$$

$$\stackrel{!}{=} (D_\mu A_\nu) B^\rho + A_\nu (\partial_\mu B^\rho + \Gamma^\rho_{\mu\lambda} B^\lambda) = \partial_\mu (A_\nu B^\rho)$$

$$\stackrel{!}{=} (D_\mu A_\nu) \underline{B^\rho} + \cancel{A_\nu (\partial_\mu B^\rho)} + A_\nu \Gamma^\rho_{\mu\lambda} B^\lambda = \partial_\mu (A_\nu B^\rho)$$

$$\stackrel{!}{=} (\partial_\mu A_\nu) \underline{B^\rho} + \cancel{A_\nu (\partial_\mu B^\rho)}$$

$$\rightsquigarrow (D_\mu A_\nu) B^\rho + A_\nu \Gamma^\rho_{\mu\lambda} B^\lambda = (\partial_\mu A_\nu) B^\rho$$

$$(D_\mu A_\nu) B^\rho = (\partial_\mu A_\nu) B^\rho - A_\nu \Gamma^\rho_{\mu\lambda} B^\lambda$$

Exer: Prove this

Tip: Consider $D_\mu V^\mu$ ⊕ use 1), 2), 3)

Covariant Derivative of $(p,q)$ tensor:

$$D_\mu T^{\nu_1 \dots \nu_p}_{\rho_1 \dots \rho_q} = \partial_\mu T^{\nu_1 \dots \nu_p}_{\rho_1 \dots \rho_q} + \Gamma^{\nu_1}_{\mu\lambda} T^{\lambda \nu_2 \dots \nu_p}_{\rho_1 \dots \rho_q} + \dots + \Gamma^{\nu_p}_{\mu\lambda} T^{\nu_1 \dots \nu_{p-1} \lambda}_{\rho_1 \dots \rho_q} -$$

$$- \Gamma^\lambda_{\mu\rho_1} T^{\nu_1 \dots \nu_q}_{\lambda \rho_2 \dots \rho_q} - \dots \Gamma^\lambda_{\mu\rho_q} T^{\lambda}_{\mu \rho_1 \dots \rho_{q-1} \lambda}$$

Proprieties of $D_\mu$

i) $D_\mu$ commutes with index contraction

ii) $D_\alpha g_{\mu\nu} = 0$

iii) $D_\mu$ commutes with raising and lowering indices

iv) $D_\mu$ commutes with scalar $D_\mu D_\nu S = D_\nu D_\mu S$

# Christoffel symbol

$$D_\mu g_{\rho\sigma} = \partial_\mu g_{\rho\sigma} - \Gamma^{\lambda}_{\mu\rho} g_{\lambda\sigma} - \Gamma^{\lambda}_{\mu\sigma} g_{\rho\lambda} \overset{(i)}{=} 0$$

$$D_\rho g_{\sigma\mu} = \partial_\rho g_{\sigma\mu} - \Gamma^{\lambda}_{\rho\sigma} g_{\lambda\mu} - \Gamma^{\lambda}_{\rho\mu} g_{\sigma\lambda} \overset{(ii)}{=} 0$$

$$D_\sigma g_{\mu\rho} = \partial_\sigma g_{\mu\rho} - \Gamma^{\lambda}_{\sigma\mu} g_{\lambda\rho} - \Gamma^{\lambda}_{\sigma\rho} g_{\mu\lambda} \overset{(iii)}{=} 0$$

$\rightarrow$ Reminders: $g_{\mu\nu} = g_{\nu\mu}$ $\rightarrow$ In GR we always consider torsion-free

So (1) − (2) − (3) $\Rightarrow$

$$\Rightarrow \boxed{\Gamma^{\lambda}_{\sigma\rho} = \frac{1}{2}\left(\partial_\rho g_{\sigma\mu} + \partial_\sigma g_{\rho\mu} - \partial_\mu g_{\rho\sigma}\right) g^{\lambda\mu}}$$

$\cdot g_{\mu\lambda} \rightarrow$

$$\Gamma_{\mu\rho\sigma} = \Gamma^{\lambda}_{\rho\sigma} g_{\lambda\mu} = \frac{1}{2}\left(\partial_\sigma g_{\mu\rho} + \partial_\rho g_{\mu\sigma} - \partial_\mu g_{\rho\sigma}\right)$$

## Parallel Transport (or Covariant Differentiation Along the curve)

Let $c: ]a,b[ \subseteq \mathbb{R} \to U \subset M$ be a parametrised curve in an open subset of $M$ which, in a given coordinate system reads $x^\mu(\lambda)$, $\lambda \in \mathbb{R}$.
We say that a vector $V$ is paralleled transported along $c$ if

$$\underbrace{\frac{DV^\mu}{d\lambda}}_{D_\lambda V^\mu} \equiv \underbrace{\frac{dx^\nu}{d\lambda}}_{\text{Tangent Vector to the curve "velocity"}} D_\nu V^\mu = 0$$

"everywhere along the curve"

The curve

Parallel Transport preserve:

Using $D_\alpha g_{\mu\nu} = 0$ we have $\overbrace{g_{\mu\nu} V^\mu V^\nu}^{V_\mu V^\nu} = \text{constant}$ "Norm of vectors"

also $g_{\mu\nu} V^\mu U^\nu = \text{const}$ "Scalar Product"

also ANGLES REMAINS CONSTANTS

- If we get the cov Derivative along the curve, allow us to define

"Covariant Acceleration"

$$a^\mu = \frac{DV^\mu}{d\lambda} = \frac{D\dot{x}^\mu}{d\lambda} =$$

$$\rightsquigarrow \dot{x}^\mu = \frac{dx^\mu}{dx}$$

$$= \ddot{x}^\mu + \Gamma^\mu_{\nu\lambda} \dot{x}^\nu \dot{x}^\mu = V^\nu D_\nu V^\mu$$

**Definition** Geodesic

$$\boxed{\frac{d^2 x^\mu}{d\lambda^2} - \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\lambda}\frac{dx^\mu}{d\lambda} = 0}$$

it's a curve OF ZERO ACCELERATION
Is what would have been a strait path in a flat space!

## LET'S SHOW THAT

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$$
$$d\tau^2 = -ds^2$$

$$\tau_{AB} = \int_A^B \sqrt{d\tau^2} = \int_A^B \sqrt{g_{\alpha\beta}\, dx^\alpha dx^\beta}$$

Consider world line $x^\alpha = x^\alpha(\sigma)$ from $\sigma = 0$ to $\sigma = 1$

$$\tau_{AB} = \int_0^1 \left[ -g_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} \right]^{1/2} d\sigma = \int_A^B L\left[ x^\alpha, \frac{dx^\alpha}{d\sigma} \right] d\sigma$$

Lets consider a Lagrangian $\longrightarrow$ $L \equiv \left[ -g_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} \right]^{1/2} = \frac{dF}{d\sigma}$

Note that $\forall\ f = f(\tau(\sigma))$ we have $\frac{dF}{d\sigma} = \frac{dF}{d\tau} \frac{d\tau}{d\sigma} = L \frac{dF}{d\tau}$ ①

## EULER LAGRANGIAN EQUATION

$$\frac{\partial L}{\partial x^\alpha} - \frac{d}{d\sigma}\left[ \frac{\partial L}{\partial (dx^\alpha / d\sigma)} \right] = 0$$

### 1° TERM

$$\frac{\partial L}{\partial x^\gamma} = -\frac{1}{2L} \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} \overset{①}{=} -\frac{L}{2} \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}$$

$\leadsto$ $\frac{dx^\alpha}{d\sigma} = L \frac{dx^\alpha}{d\tau}$

### 2° TERM

$$\frac{\partial L}{\partial (dx^\gamma / d\sigma)} = -\frac{1}{2L} g_{\alpha\beta} \left( \frac{dx^\beta}{d\sigma} \delta_{\alpha\gamma} + \frac{dx^\alpha}{d\sigma} \delta_{\beta\gamma} \right)$$

$\delta_{\alpha\gamma} = \frac{\partial x^\alpha}{\partial x^\gamma}$

$$= -\frac{1}{2L} \left( g_{\gamma\beta} \frac{dx^\beta}{d\sigma} + g_{\alpha\gamma} \frac{dx^\alpha}{d\sigma} \right)$$

$\leadsto$ Metric is simmetric

$$= -\frac{1}{L} g_{\alpha\gamma} \frac{dx^\alpha}{d\sigma}$$

## Derivation of II° Term

$$\frac{d}{d\sigma}\left[\frac{\partial L}{\partial(dx^\gamma/d\sigma)}\right] = \frac{d}{d\sigma}\left[-\frac{1}{L}\, g_{\alpha\gamma}\frac{dx^\alpha}{d\sigma}\right]$$

$$\overset{(1)}{=} L\frac{d}{d\tau}\left[-\frac{1}{\cancel{L}}\, g_{\alpha\gamma}\cancel{L}\frac{dx^\alpha}{d\tau}\right]$$

$$= L\left[g_{\alpha\gamma}\frac{d^2}{d\tau^2}x^\alpha + \frac{dg_{\alpha\gamma}}{d\tau}\frac{d}{d\tau}x^\alpha\right]$$

$$= L\left[g_{\alpha\gamma}\frac{d^2x^\alpha}{d\tau^2} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta}\frac{\partial x^\beta}{d\tau}\frac{dx^\alpha}{d\tau}\right]$$

$$= L\left[g_{\alpha\gamma}\frac{d^2x^\alpha}{d\tau^2} + \frac{1}{2}\left[\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial g_{\gamma\beta}}{\partial x^\alpha}\right]\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\right]$$

## Lagrange Equation:

$$\frac{\partial L}{\partial x^\alpha} - \frac{d}{d\sigma}\left[\frac{\partial L}{\partial(dx^\alpha/d\sigma)}\right] = L\left[g_{\alpha\gamma}\frac{d^2x^\alpha}{d\tau^2} + \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial g_{\gamma\beta}}{\partial x^\alpha}\right)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\right] - \frac{L}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

$$\Rightarrow g_{\alpha\gamma}\frac{d^2x^\alpha}{d\tau^2} = -\frac{1}{2}\left[\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial g_{\gamma\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right]\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} =$$

$$= -\frac{1}{2}\left[\frac{\partial g_{\delta\gamma}}{\partial x^\beta} + \frac{\partial g_{\gamma\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right]\frac{dx^\delta}{d\tau}\frac{dx^\beta}{d\tau} =$$

$$= -\frac{1}{2}\, g_{\alpha\gamma}\,\Gamma^\alpha_{\delta\beta}\frac{dx^\delta}{d\tau}\frac{dx^\beta}{d\tau}$$

$$\Rightarrow \boxed{\frac{d^2x^\alpha}{d\tau^2} = -\frac{1}{2}\Gamma^\alpha_{\delta\beta}\frac{dx^\delta}{d\tau}\frac{dx^\beta}{d\tau}} \qquad \square$$

If we define $L_\perp = \frac{L^2}{2}$ it's been shown that also satisfies Eul-Lag.

# Geodesics of Flat S-T in cartesian Polar Coordinates

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\varphi^2$$

i) Cartesian $g_{xx} = 1 \quad g_{xy} = 0 = g_{yx}$ | $g^{xx} = 1 \quad g^{xy} = 0 = g^{yx}$

$g_{yy} = 1$ | $g^{yy} = 1$

2) Polar $g_{rr} = 1 \quad g_{r\varphi} = 0 = g_{\varphi r}$ | $g^{rr} = 1 \quad g^{r\varphi} = 0 = g^{\varphi r}$

$g_{\varphi\varphi} = r^2$ | $g^{\varphi\varphi} = r^{-2}$

## Geodesic equation:

1) Cartesian

All $\Gamma$ symbols are zero $\quad (g_{\mu\nu} = \delta_{\mu\nu} \rightarrow \text{All } \partial_\xi g_{\mu\nu} = 0)$

$$\rightarrow \frac{d^2x}{d\lambda^2} = \frac{d^2y}{d\lambda^2} = 0$$

2) Polar

$$\partial_r g_{rr} = 0 \qquad \partial_\varphi g_{\mu\nu} = 0 \quad \forall\, \mu,\nu = r,\phi$$
$$\partial_r g_{r\varphi} = 0$$
$$\partial_r g_{\varphi\varphi} = 2r$$

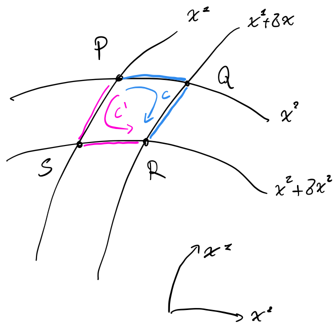
$$\Rightarrow \Gamma^r_{\varphi\varphi} = -r \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r} \qquad \Gamma^\varphi_{\varphi r} = \frac{1}{r} \quad \text{(symmetric)}$$

$$\rightsquigarrow \frac{d^2r}{d\lambda^2} + \Gamma^r_{\varphi\varphi}\left(\frac{d\varphi}{d\lambda}\right)^2 = 0 \quad \Rightarrow \quad \ddot{r} - r\dot{\varphi}^2 = 0$$

$$\frac{d^2\varphi}{d\lambda^2} + 2\Gamma^\varphi_{r\varphi}\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda} = 0 \quad \Rightarrow \quad \ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} = 0$$

Even in Flat Space There are $\neq 0$ Christoffel Symbols

# Transport Vectors on curved space




Remind: $D_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\nu_{\mu\lambda} V^\lambda$ Covariant Derivative

$\frac{dx^\nu}{d\lambda} D_\nu V^\nu = 0$ Parallel Transport

## i) Parallel Transport along P → Q

Only $x^1$ changes

$$\frac{dx^\nu}{d\lambda} D_\nu V^\mu = \frac{dx^1}{d\lambda} D_1 V^\mu + \underbrace{\cancel{\frac{dx^2}{d\lambda}}}_{=0} D_2 V^\mu = 0$$

So $\frac{dx^1}{d\lambda} D_1 V^\mu = 0 = \frac{dx^1}{d\lambda}\left[\partial_1 V^\mu + \Gamma^\mu_{1\lambda} V^\lambda\right]\Big|_{x^2\,\text{constant}}$

$$\Rightarrow \frac{\partial V^\mu}{\partial x^1}\Big|_{x_2} = -\Gamma^\mu_{1\lambda} V^\lambda \quad \Rightarrow \int_{x^1}^{x^1+\delta x^1} \frac{\partial V^\mu}{\partial x^1} dx^1 = -\int \Gamma^\mu_{1\lambda} V^\lambda dx^1\Big|_{x^2}$$

$$\Rightarrow V_c^\mu(Q) - V^\mu(P) = -\int \Gamma^\mu_{1\lambda} V^\lambda dx^1\Big|_{x^2}$$

$$\Rightarrow V_c^\mu(Q) = V^\mu(P) - \underbrace{\int \Gamma^\mu_{1\lambda} V^\lambda dx^1\Big|_{x^2}}_{\equiv I_1} \qquad (3)$$

## ii) Parallel transport Q → R, $x^1$ is constant

Like before $\frac{dx^2}{d\lambda} D_2 V^\mu = 0 \Rightarrow \frac{dx^2}{d\lambda}\left[\partial_2 V^\mu + \Gamma^\mu_{2\lambda} V^\lambda\right] = 0$

$$\ldots \Longrightarrow V_c^\mu(R) = V_c^\mu(Q) - \underbrace{\int_{x_2}^{x_2+\delta x_2} \Gamma^\mu_{2\lambda} V^\lambda dx_2 \Big|_{x^1+\delta x^1}}_{\equiv I_2} \qquad (4)$$

iii) Parallel Transport $P \to S$

$$V_{c'}^{\mu}(S) = V_c^{\mu}(P) - \underbrace{\int_{x^2}^{x^2+\delta x^2} \Gamma^{\mu}_{2\lambda} V^{\lambda} dx^2 \Big|_{x^1}}_{\equiv I_3} \qquad (5)$$

iv) Parallel Transport $S \to R$

$$V_{c'}^{\mu}(R) = V_{c'}^{\mu}(S) - \underbrace{\int_{x^1}^{x_1+\delta x_1} \Gamma^{\mu}_{1\lambda} V^{\lambda} dx^1 \Big|_{x^2+\delta x^2}}_{\equiv I_4} \qquad (6)$$

We want to calculate differences

$$\begin{aligned} V_c^{\mu}(R) - V_{c'}^{\mu}(R) &= V_c^{\mu}(Q) - I_2 - V_{c'}^{\mu}(S) + I_4 = \\ &\overset{!}{=} V^{\mu}\cancel{(P)} - I_1 - I_2 - V^{\mu}\cancel{(P)} + I_3 + I_4 \\ &\overset{!}{=} I_3 - I_1 + I_4 - I_2 \\ &\overset{!}{=} \underbrace{\int_{x^1}^{x^1+\delta x^1} \left( \Gamma^{\mu}_{1\lambda} V^{\lambda}\Big|_{x^2+\delta x^2} - \Gamma^{\mu}_{1\lambda} V^{\lambda} \right) dx^1}_{I_{41}} + \underbrace{\int_{x^2}^{x^2+\delta x^2} \left( \Gamma^{\mu}_{2\lambda} V^{\lambda}\Big|_{x^1} - \Gamma^{\mu}_{2\lambda} V^{\lambda}\Big|_{x^1+\delta x^1} \right) dx^2}_{I_{32}} \end{aligned}$$

In the limit $\delta x^2$ and $\delta x^1$ are small we can Taylor expand integrals

$$\Gamma^{\mu}_{1\lambda} V^{\lambda}\Big|_{x^2+\delta x^2} \simeq \Gamma^{\mu}_{1\lambda} V^{\lambda}\Big|_{x^2} + \left[\frac{\partial}{\partial x^2} \left(\Gamma^{\mu}_{2\lambda} V^{\lambda}\right)\right]_{x^2} \delta x^2$$

$$\Gamma^{\mu}_{2\lambda} V^{\lambda}\Big|_{x^1+\delta x^1} \simeq \Gamma^{\mu}_{2\lambda} V^{\lambda}\Big|_{x^1} + \left[\frac{\partial}{\partial x^1} \left(\Gamma^{\mu}_{2\lambda} V^{\lambda}\right)\right]_{x^1} \delta x^1$$

So

$$I_{41} = \delta x^2 \int_{x^2}^{x^2+\delta x^2} \left[\frac{\partial}{\partial x^2} \Gamma^{\mu}_{1\lambda} V^{\lambda}\right]_{x^2} dx^1$$

$$I_{32} = -\delta x^1 \int_{x^2}^{x^2+\delta x^2} \left[\frac{\partial}{\partial x^1} \Gamma^{\mu}_{2\lambda} V^{\lambda}\right]_{x^2} dx^2$$

For small $\delta x^{1,2}$ we consider the integral to be constant ...

$$I_{41} \simeq \delta x^2 \delta x^1 \left[\partial_2\left(\underline{\Gamma^\mu_{2\lambda} V^\lambda}\right)\right]_{x^2}$$

$$I_{32} \simeq -\delta x^1 \delta x^2 \left[\partial_1\left(\Gamma^\mu_{1\lambda} V^\lambda\right)\right]_{x^1}$$

Open up the parhentesis

$$I_{32} = -\delta x^1 \delta x^2 \left(\underline{(\partial_1 \Gamma^\mu_{2\lambda}) V^\lambda + \Gamma^\mu_{2\lambda}\partial_1 V^\lambda}\right)$$

Using $\frac{dV^\mu}{d\lambda} = -\Gamma^\mu_{\lambda\alpha} V^\alpha$ ...

$$= -\delta x^1 \delta x^2 \left(\partial_1 \Gamma^\mu_{2\lambda} V^\lambda + \Gamma^\mu_{2\lambda}(-\Gamma^\lambda_{1\alpha} V^\alpha)\right)$$

$$= -\delta x^1 \delta x^2 \left(\partial_1 \Gamma^\mu_{2\lambda} - \Gamma^\alpha_{1\lambda}\Gamma^\lambda_{1\alpha}\right) V^\lambda$$

swapped $\alpha$ and $\lambda$

Similarly

$$I_{41} = \delta x^1 \delta x^2 \left(\partial_2 \Gamma^\mu_{1\lambda} - \Gamma^\alpha_{2\lambda}\Gamma^\mu_{1\alpha}\right) V^\lambda$$

In the end:

$$V^\mu_c(R) - V^\mu_{c'}(R) = \delta x^2 \delta x^2 \left(\partial_1 \Gamma^\mu_{\lambda 2} - \partial_2 \Gamma^\mu_{\lambda 1} + \Gamma^\rho_{\lambda 2}\Gamma^\mu_{\rho 1} - \Gamma^\rho_{\lambda 1}\Gamma^\mu_{2\rho}\right) V^\lambda$$

But nothing special in 1 or 2 ... $1 \to \sigma$, $2 \to \nu$

$$V^\lambda_c(R) - V^\lambda_{c'}(R) = \delta x^\sigma \delta x^\nu R^\lambda{}_{\mu\sigma\nu}(P)\, V^\mu(P)$$

Where $R^\lambda{}_{\mu\sigma\nu}(P)$ is the Riemann Tensor

$$\boxed{R^\lambda{}_{\mu\sigma\nu}(P) = \partial_\sigma \Gamma^\mu_{\lambda\nu} - \partial_\nu \Gamma^\mu_{\lambda\sigma} + \Gamma^\rho_{\lambda\nu}\Gamma^\mu_{\rho\sigma} - \Gamma^\rho_{\lambda\sigma}\Gamma^\mu_{\nu\rho}}$$

Properties

0) Is a tensor

1) $R^\mu{}_{\nu\rho\sigma} = -R^\mu{}_{\nu\sigma\rho}$

2) $R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}$ where $R_{\mu\nu\rho\sigma} = g_{\mu\alpha} R^\alpha{}_{\nu\rho\sigma}$

3) $R^\mu{}_{\nu\rho\sigma} + R^\mu{}_{\sigma\nu\rho} + R^\mu{}_{\rho\sigma\nu} = 0$
$R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\rho\sigma\nu} = 0$

4) $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$

5) Bianchi identity

$$D_\lambda R^\mu{}_{\nu\rho\sigma} + D_\rho R^\mu{}_{\nu\sigma\lambda} + D_\sigma R^\mu{}_{\nu\lambda\rho} = 0$$

**Definition:** Ricci Tensor

$$R_{\mu\nu} \equiv g^{\rho\lambda} R_{\mu\rho\nu\lambda} = R^{\rho}{}_{\mu\rho\nu}$$

Properties

- Symmetric, 10 indept components

**Definition:** Ricci Scalar

$$R = g_{\mu\nu} R^{\mu\nu}$$

---

## Einstein Tensor

Bianchi $\quad D_\lambda R_{\mu\nu\rho\sigma} + D_\rho R_{\mu\nu\sigma\lambda} + D_\sigma R_{\mu\nu\lambda\rho} = 0$

Multiply times $g^{\nu\sigma} g^{\mu\rho}$

$$g^{\nu\sigma} g^{\mu\rho} \left( D_\lambda R_{\mu\nu\rho\sigma} + D_\rho \underbrace{R_{\mu\nu\sigma\lambda}}_{-\nu\mu\sigma\lambda} + D_\sigma \underbrace{R_{\mu\nu\lambda\rho}}_{-\mu\nu\rho\lambda} \right) = 0$$

$$\Rightarrow g^{\nu\sigma} D_\lambda R_{\nu\sigma} - g^{\mu\rho} D_\rho R_{\mu\lambda} - g^{\nu\sigma} D_\sigma R_{\nu\lambda} = 0$$

$$\Rightarrow D_\lambda R - D^\mu R_{\mu\lambda} - D^\nu R_{\nu\lambda} = 0$$

$$\Rightarrow \underbrace{D_\lambda R}_{g_{\lambda\sigma} D^\sigma R} - 2 D^\sigma R_{\sigma\lambda} = 0$$

So $\quad D^\sigma \underbrace{\left[ R_{\lambda\sigma} - \frac{1}{2} g_{\lambda\sigma} R \right]}_{G_{\lambda\sigma}} = 0$

- Covariantly Conserved $\quad D^\sigma G_{\lambda\sigma} = 0$
- It's true also for $g_{\mu\nu}$ $\quad D^\mu g_{\mu\nu} = 0$ $\Big\}$ $\quad D^\sigma \left[ G_{\lambda\sigma} + \Lambda g_{\mu\nu} \right] = 0$

$\Lambda$ → Cosmological Constant

Dark Matter

## Geodesic Deviation

- $\mathcal{L}_0$ parametrized by $\lambda$

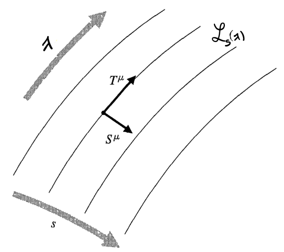
$$T^\mu = \frac{dx^\mu}{d\lambda} \rightarrow \text{tangent vector}$$

- Consider $\{\mathcal{L}\}$ Family of geodesics, parametrized by $\lambda$

  which geodesic parametrised by $s$

$s$ $\downarrow$ which geodesic

$\lambda$ $\downarrow$ where on geodesic




- If $X^\mu(s,\lambda)$ are the coordinates of geodesics in $\{\mathcal{L}\}$, the tangent vector

$$T^\mu = \frac{\partial x^\mu(s,\lambda)}{\partial \lambda}$$

N.B. if $\lambda$ is a (proper time), then $T :=$ velocity

- We can define a "Deviation Vector"

$$X^\mu \equiv \frac{\partial x^\mu(s,\lambda)}{\partial s}$$

$\rightarrow$ define some sort of velocity between geodesics

- "Relative Acceleration"

$$A^\mu \equiv T^\alpha D_\alpha (T^\beta D_\beta X^\mu)$$

Express: The relative acceleration between 2 neighboring geodesics is proportional to the curvature

- One can show that $\boxed{A^\mu = R^\mu{}_{\nu\beta\sigma} T^\nu T^\mu X^\sigma}$

- In weak Field limit

$$g_{00} = 1 + \frac{2\,\Phi(x,y,z)}{c^2}$$ → gravitational potential

Acceleration involves $\frac{\partial^2 \Phi}{\partial x^i \partial x^j}$ } Tidal Force Tensor $\Delta_{ij}$

In empty space $\Delta^2 \Phi = 0 \leftrightarrow \sum_{i=1}^{3} \Delta_{ii} = 0$

$$R^{\mu}{}_{\alpha\mu\beta}\, \dot{x}^{\alpha} \dot{x}^{\beta} = 0$$

Newtonian, empty space

$$\nabla^2 \Phi = 0$$

Newtonian, empty space, introduce Matter

$$\nabla^2 \Phi = \text{const} \cdot \rho$$

$\rightsquigarrow$ Here we are trying to obtain the same equation but using tensor

---

PARENTHESIS

1) Volume Element

$d^4x \rightarrow$ is not invariant under coordinate transformation

It transforms like this: $\quad d^4x' = \det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right) d^4x$

Metric Tensor Transform:

$$g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma} \quad \Longrightarrow \quad \det(g'_{\mu\nu}) = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \det(g_{\rho\sigma})$$

with this Knowledge we conclude that

$$d^4x = dx^1 \wedge \ldots dx^n = \sqrt{\frac{\det[g]}{\det[g]}} \frac{\partial(x)}{\partial(x)} dx^1 \wedge \ldots dx^n = d^4x'$$

e that:

$$\sqrt{\det[g]} \cdot dx^1 \wedge \ldots dx^n = \sqrt{\det[g]} \cdot dx^1 \wedge \ldots dx^n$$

Since $\quad dV = \sqrt{\det(g_{\mu\nu})}\, d^4x \quad \Longrightarrow \quad dV' = dV$

3) GAUSS THEOREM

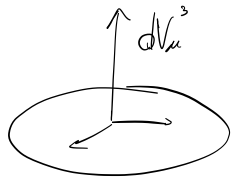
$$\int_S F = \int_V \partial F$$

$$\int d^4x \sqrt{g_4}\, D_\mu A^\mu = \int_{\partial V} d^3x \sqrt{g_3}\, D_\mu A^\mu$$

## 1) Density

In 3D space a "density" $\rho$: $N = \int \rho \, dV^3$

$$\underbrace{dV^4}_{(3+1)\,D} = dx^\mu \underbrace{dV^3_\mu}_{3D}$$

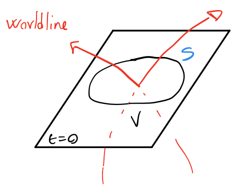



### Example 1

Volume element is S-T slice whit $t=0$

$$dV_\mu = (dV^3, 0, 0, 0) \longrightarrow dN = \underbrace{\rho^\mu}_{\text{3D density}} \underbrace{dV_\mu}_{\text{3D volume}}$$ # particle in 3D volume

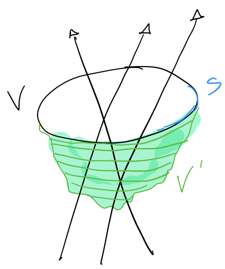
### Example 2 PARTICLE NUMBER DENSITY




⇝ Volume $V$ delimited in a surface $S$

$$dN = n^\mu \, dV_\mu$$ ⇝ number of worldlines (ake POINTS - PARTICLE at $t=0$)

$$N = \int_V \underbrace{n^\mu}_{\text{Number density vector}} dV_\mu$$




# PARTICLE is a phisycs object ⇒ Do not Change

Consequence: $V$ is disc, $V'$ is a surface ⇒ # particle that cross $V$, always cross $V'$

⇒ IF $V$ and $V'$ define an Hypervolume $H$, by Gauss Theorem

$$\int_H \partial_\mu n^\mu d^4H = 0 \quad \forall V, V' \implies \underline{\underline{\partial_\mu n^\mu = 0}}$$

IF we were in Minkowsky space $\boxed{\partial_t n^t = -\vec{\nabla}\cdot\vec{n}}$ continuity relation

We could also write $n^\mu(x) = \underbrace{n(x)}_{\text{scalar density}} V^\mu(x)$

⇝ In the same way we can think about other quantities

# Stress - Energy - Momentum Tensor

Particle at velocity $V^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau}$ carries a 4-momentum $p^\mu = m\gamma(1, v^x, v^y, v^z)$

Question? Total momentum through a surface S?

→ To count # particle in a spacetime surface we needed a Number-flux 4-vector.

Answer: $P^\mu = \int T^{\mu\nu} dV_\nu$ ⇒ Now we need a Tensor

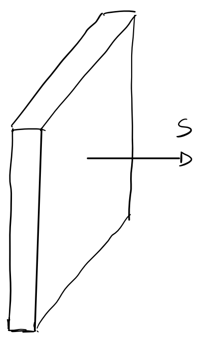
---

Let's consider an ensemble of particle

$$T^{\mu\nu} \equiv n p^\mu v^\nu = \underbrace{nm}_{\rho(x)} V^\mu V^\nu$$

: Flow of $\mu$ momentum going through a unit box of constant $\nu$

- $T^{00}$ = Energy density ( )
- $T^{0i}$ = Energy density Flow in direction $i$
- $T^{i0}$ = Momentum Density
- $T^{ij}$ = Flow of $i$-th momentum in $j$-th direction




$$dp^i = \underbrace{(\vec{V}\cdot dt)\cdot d\vec{S}\, n}_{\#\text{ particle leave the box}} P^i$$

Let $i=j$ ⇒ $dp^i > 0$ → Momentum transfered to surface S

⇒ $F^i > 0$ ⇒ pressure $P_{in} > 0$

⇒ $T^{ii} > 0$ ⇒ pressure in the box

## Conservation law

Assume that all particles have same velocity travel along geodesics

① $$a^\mu = \frac{dV^\mu}{d\tau} = \frac{dx^\nu}{d\tau} D_\nu V^\mu(x) = 0$$

② Particle don't appear/disappear : $D_\mu \left( n(x) V^\mu(x) \right) = 0$

$$D_\mu T^{\mu\nu} = D_\mu \left( m n(x) V^\mu V^\nu \right) = m V^\nu \underbrace{D_\mu \left( n(x) V^\mu(x) \right)}_{=0\ (2)} + \underbrace{m n(x) V^\mu D_\mu V^\nu(x)}_{=0\ (1)}$$

$$\Longrightarrow \boxed{D_\mu T^{\mu\nu} = 0}$$

## Special Cases

### 1| IMMOBILE DUST

↳ set of particles w/ worldlines

$T^{tt} = \rho_0 \longrightarrow$ Rest mass density

### 2| GENERIC DUST

$\longrightarrow$ For observer moving along with the dust $V^\mu = (1,0,0,0) \Rightarrow T^{\mu\nu} = \rho_0 V^\mu V^\nu$

$$V = \gamma \left(1, V^x, V^y, V^z\right) \Longrightarrow T^{\mu\nu} = \rho \cdot \gamma^2 \begin{pmatrix} 1 & V_x & V_y & V_z \\ V_x & V_x^2 & V_x V_y & V_x V_z \\ V_y & & V_y^2 & \\ V_z & & & V_z^2 \end{pmatrix}$$

$\longrightarrow$ Non zero stress, mass current

### 3| STATIC MONOENERGETIC GAS

Same V, random orientation

$V^x V^y$ etc $= 0$ when diverged

$$(V^i)^2 = \frac{V^2}{3}$$

$$\Longrightarrow T^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & \frac{V^2}{3} & & 0 \\ & & \frac{V^2}{3} & \\ 0 & & & \frac{V^2}{3} \end{pmatrix}$$

## 4| STATIC IDEAL GAS

$V \rightarrow$ Follows statistical distribution $\qquad T^{\mu\nu} = \begin{pmatrix} \rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix}$

$$V^{\mu} = (1,0,0,0) \quad \text{and} \quad g^{\mu\nu} = \text{diag}\,(1,-1,-1,-1)$$

$$T^{\mu\nu} = (\rho + P)\, V^{\mu} V^{\nu} - \rho\, g^{\mu\nu}$$

$\rightarrow$ Also valid for observer traversing the gas at some velocity

## 4.b| Radiation $\quad P_R = \frac{\rho_R}{3}$

$$T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & \rho/3 & & \\ & & \rho/3 & \\ & & & \rho/3 \end{pmatrix}$$

# Gravitation

We are trying to obtain

$$\frac{d\vec{x}}{dt^2} = -\vec{\nabla}\Phi \qquad \text{eg: } \Phi = \frac{G\nu M}{r}$$

but in CURVED space-time.

Let's define the Newtonian limit

There are various requirements

(metric is a perturbation of flat space)

1) Weak Field $\quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \oplus$ keep only terms up to linear in $h_{\mu\nu}$ + derivatives

↳ small perturbation

Note: statement about field + coordinates

a) (The metric $-+++$)

b) $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$

$$(I+A)^{-1} = I^{-1} - I^{-1} A I^{-1} + O(A^2)$$

2) Small velocities: $\quad \frac{dx^i}{dt} \ll 1 \longleftrightarrow \frac{dx^i}{d\tau} \ll \frac{dt}{d\tau}$ (Moving Slow)

3) Stationary Fields: $\quad \partial_0 g_{\mu\nu} = 0 \Leftrightarrow \partial_0 h_{\mu\nu} = 0$ (Unchanging with time)

Our goal is to go from the equation of motion in General Relativity (Geodesis) to the Newtonian equation of motion

Geodesis

$$\frac{d^2x}{d\tau^2} + \Gamma^{\mu}_{\nu\lambda} \frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0$$

And Recall

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2}\left(\partial_\lambda g_{\nu\alpha} + \partial_\nu g_{\lambda\alpha} - \partial_\alpha g_{\lambda\nu}\right) g^{\mu\alpha}$$

THE weak field condition we should

$g^{\mu\alpha} = \eta^{\mu\alpha} + o(h)$

why? only 1° order in $h$ are allowed and $(\partial(\eta+h))g$, $g \equiv \eta$

$\rightsquigarrow \partial\eta_{\mu\nu} = 0$

Weak Field $\Rightarrow \quad \Gamma^{\mu}_{\nu\lambda} = \frac{1}{2}\left(\partial_\lambda h_{\nu\alpha} + \partial_\nu h_{\lambda\alpha} - \partial_\alpha h_{\lambda\nu}\right)\eta^{\mu\alpha}$

Going Back on geodesis, using SMALL VELOCITY

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{00} \frac{dt}{d\tau}\frac{dt}{d\tau} = 0 \quad \leftarrow \text{The terms } \Gamma^\mu_{ij} \frac{dx^i}{d\tau}\frac{dx^j}{d\tau} \simeq 0$$

This means that we only have to compute $\Gamma^\mu_{00}$

$$\Gamma^\mu_{00} = \frac{1}{2} \underbrace{g^{\mu\nu}}_{\eta^{\mu\nu} + o(h)} \big( \underbrace{\cancel{\partial_0 g_{\nu 0}} + \cancel{\partial_0 g_{\nu 0}}}_{\text{Static condition}} - \partial_\nu g_{00} \big) \implies \Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu\nu}\partial_\nu h_{00}$$

$$= -\frac{1}{2}\eta^{\mu i}\partial_i h_{00} \quad \text{Static Condition}$$

$i = 1,2,3$ NO 0!

So $\Gamma^0_{00} = 0 \qquad \Gamma^i_{00} = -\frac{1}{2}\partial^i h_{00}$

So the geodesics equation

$$\boxed{\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{00}\frac{dt}{d\tau}\frac{dt}{d\tau} = 0}$$

Splitting Temporal – Spatial Part:

Time $\frac{d^2 t}{d\tau^2} = 0$ ①

Space $\frac{d^2 x^i}{d\tau^2} = \frac{1}{2}\partial h_{00}\left(\frac{dt}{d\tau}\right)^2$ ②

① Solution: $t = a\tau + b \implies \underbrace{\frac{d^2 x}{d\tau^2} = \frac{d^2 x}{dt^2}\left(\frac{dt}{d\tau}\right)^2}$

$\rightarrow \frac{dt}{d\tau}$ is constant

Replace in ②, delete $\left(\frac{dt}{d\tau}\right)^2$

$$\implies \boxed{\frac{d^2 x^i}{dt^2} = \frac{1}{2}\partial h_{00}}$$

And we've done because $\frac{d^2\vec{x}}{dt^2} = -\vec{\nabla}\Phi \quad \xrightarrow{\text{Tensorial Way}} \quad \frac{d^2 x^i}{dt^2} = -\partial^i\Phi$

Which that we can identify $h_{00} = -2\Phi \rightarrow g_{00} = -(1+2\Phi)$

Newton Gravity can be reproduced by using the metric

$$ds^2 = -(1+2\Phi(\vec{x}))\,dt^2 + d\vec{x}^2$$

Therefore, we have shown that the curvature of spacetime is indeed sufficient to describe gravity in the Newtonian limit, as long as the metric takes the form

# The Einstein Field Equations

Newtonian mechanics $\quad \nabla^2 \Phi = 4\pi G_N \underbrace{\rho}_{\text{mass density}} \quad ①$

Then we discovered $\quad g_{00} = -(1+2\Phi) \quad ②$

Identity $\quad T_{00} = \rho \Rightarrow \nabla^2 g_{00} = -8\pi G_N T_{00}$

Search Tensor Relation $\quad E_{\mu\nu} = 8\pi G_N T_{\mu\nu}$

What is $E_{\mu\nu}$?

Sensible requirements:

1) $E_{\mu\nu}$ is tensor

2) Must involve either second derivative or First derivative squared

3) $E_{\mu\nu}$ is simmetric

4) Covariant Conservation $\Rightarrow D_\mu E^{\mu\nu} = 0$

5) In the Newtonian Limit we must have $E_{00} = -\nabla^2 g_{00}$

---

1) + 2) $\Rightarrow \quad R_{\mu\nu} = \underbrace{a R_{\mu\nu} + b g_{\mu\nu} R}_{\text{3) also satisfied}}$ (we are trying with this)

4) $\Rightarrow$ Rewrite as $\quad E_{\mu\nu} = a G_{\mu\nu} + c g_{\mu\nu} R \quad$ (Remember $D^\mu g_{\mu\nu} = 0$)

For $D^\mu E_{\mu\nu} = 0$, we must have that $D^\mu (c g_{\mu\nu} R) = c (D_\nu R) = 0$ so we have

2 solutions: $D_\nu R = 0$ ③ ; $c = 0$ ④

③ Physically inconsistent $\quad g^{\mu\nu} E_{\mu\nu} = \underbrace{g^{\mu\nu} a G_{\mu\nu}}_{aR - 2aR} + c \underbrace{g^{\mu\nu} g_{\mu\nu}}_{=\delta^\mu_\mu = 4} R = \underbrace{(4c - a) R}_{\text{constant if ③}}$

because $E_\mu^\nu = \kappa T^{\mu\nu}$

The $T$ should describe generic distribution $\Rightarrow$ cannot be const

So $E_{\mu\nu} = a\, G_{\mu\nu}$ $\to$ To Find $a$ go to Newtonian Limit

1) Weak Field $\to$ (Change from previous)
Keep terms up to linear in $h$, with $g_{\mu\nu} = \eta + h$

2) Time independent metric: $\partial_0 g_{\mu\nu} = \partial_0 h_{\mu\nu} = 0$

3) Non relativistic source: $T_{00} = \rho$, $T_{\mu\nu} = 0$ otherwise $\to$
$\to$ Only need $E_{00} = a\, G_{00}$

$$G_{00} = R_{00} - \frac{1}{2} g_{00} \underbrace{R}_{\substack{\text{at least linear in } h \\ \text{weak field}}} \Rightarrow g_{00} \to \eta_{00} = -1 \to G_{00} \simeq R_{00} + \frac{1}{2} R \quad (5)$$

$$T_{ij} = 0 \Rightarrow G_{ij} = 0 \Leftrightarrow R_{ij} = \frac{1}{2} \underbrace{g_{ij}}_{\text{just Mink. part}} R \Rightarrow R_{ij} = \frac{1}{2} \delta_{ij} R \quad (6)$$

$$(5), (6) \Rightarrow R = g^{\mu\nu} R_{\mu\nu} = \eta^{\mu\nu} R_{\mu\nu} = -R_{00} + \frac{3}{2} R \Rightarrow R = 2 R_{00}$$

$$\Rightarrow G_{00} = R_{00} + \frac{1}{2} R = 2 R_{00}$$

$$R_{00} = R^{\mu}{}_{0\mu0}$$

Exercise:
Show that for time independent fields in the weak limit $R_{00} = -\frac{1}{2} \nabla^2 g_{00}$

All in all: $G_{00} = 2R_{00} = -\nabla^2 g_{00}$

For (v) to hold $\Rightarrow E_{00} = -\nabla^2 g_{00} \Leftrightarrow a\, G_{00} = -\nabla^2 g_{00} \Leftrightarrow a = 1$

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu}}$$ Einstein Equation

$\leadsto$ Tell us how the curvature of space time reacts to the presence of energy momentum.

Used for calculate the metric tensor $\longrightarrow$ Calculate Christoffel symbols $\longrightarrow$ Determine how objects move

# Schwarzchild Solution

Sferically symmetric Gravitational Field.

⤳ Relevant situation to describe the Field created by Earth or Sun

In this chapter: Simple case of vacuum solution with perfect SPHERICAL SYMMETRY

Recall Eistein Equation $G_{\mu\nu} = 0$, let's work in Schwarzschild coordinates $(t, r, \theta, \varphi)$

From movements of particles we can try to know how space is CURVED and therefore what is the distribution of mass

The Swch coordinates are:

i) $(1+3)$D

ii) Of spherical symmetry

iii) Static

From ii) and iii)

$$\left.\begin{aligned} (t, r, \theta, \varphi) &\longrightarrow (t, r, -\theta, \varphi) \quad \Rightarrow g'_{\mu\nu} = g_{\mu\nu} \\ (t, r, \theta, \varphi) &\longrightarrow (t, r, \theta, -\varphi) \quad \Rightarrow g'_{\mu\nu} = g_{\mu\nu} \end{aligned}\right\} \text{From ii)}$$

$$(t, r, \theta, \varphi) \longrightarrow (-t, r, \theta, \varphi) \quad \Rightarrow g'_{\mu\nu} = g_{\mu\nu} \quad \} \text{ From iii)}$$

But $g_{\mu\nu}$ is a $(0,2)$ tensor, so we can write

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\mu} g_{\alpha\beta} \qquad \text{General Transfor. Law}$$

Then as an example

$$\left.\begin{aligned} g'_{11} &= \frac{\partial x^\alpha}{\partial(-t)} \frac{\partial x^\beta}{\partial(-t)} g_{\alpha\beta} = g_{11} \\ g'_{12} &= \frac{\partial x^\alpha}{\partial(-t)} \frac{\partial x^\beta}{\partial(r)} g_{\alpha\beta} = -g_{12} \end{aligned}\right\} \quad g'_{1\nu} = -g_{1\nu}$$

But $\underbrace{g'_{\mu\nu} = g_{\mu\nu}}_{\text{Requirement}} \Rightarrow \boxed{g_{1\nu} = 0 \ \forall \nu \neq 1}$

Similarly $g_{2\nu}=0 \quad \forall \nu \neq 2 \qquad g_{3\nu}=0 \quad \forall \nu \neq 3 \qquad g_{\mu\nu}=0 \quad \forall \mu \neq \nu$

$\Rightarrow$ $g_{\mu\nu}$ is DIAGONAL (in this coord. system)

iv) Also implies that on each radial line $(\theta, \varphi = \text{const})$, $g_{tt}$ and $g_{rt}$ can only depend on $t, r$.
Adding iii) we can conclude that $g_{tt}, g_{rr}$ only depend on $r$.

$\Rightarrow$ On each hyper surface of $r=\text{const}$ the metric must be the same as the one on a sphere:

$$c^2 d\tau^2 = A(r)\, dt^2 - B(r)\, dr^2 - r^2 (\underbrace{d\theta^2 + \sin^2\theta\, d\varphi^2}_{\text{surface of sphere}})$$

Now we want $A(r)$, $B(r)$

---

Because $dt$ and $dr$ are multiplied by the functions, we don't have neither proper time or proper distance in the line element

---

Note that

$$G_{\mu\nu}=0 \Rightarrow g^{\mu\nu} G_{\mu\nu}=0 \Rightarrow g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) = 0 \Rightarrow R - \frac{1}{2} 4R = 0$$
$$\Rightarrow -R = 0 \Rightarrow R = 0$$

Called Ricci Flat Manifold

The steps to solve the equation are:

1) Christoffel symbols
2) Computing $R^{\alpha}{}_{\nu\beta\sigma}$
3) Computing $R_{\mu\nu}$
4) Solve $R_{\mu\nu}=0$

## CHRISTOFFEL SYMBOLS

2 ways — 1: Brute Force
— 2: More Practical

Consider 3 THINGS:

1) Geodesic Equation

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\nu\beta} \frac{dx^\nu}{d\tau} \frac{dx^\beta}{d\tau} = 0$$

2) Lagrangian

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

3) Euler - Lagrange Equations

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial (\dot{x}^\alpha)} - \frac{\partial \mathcal{L}}{\partial x^\alpha} = 0 \quad \text{where} \quad \dot{x} \equiv \frac{dx}{d\tau}$$

We can derive ① from ② and ③

During this derivation we will encounter

$$\frac{d}{d\tau}\left[ g_{\alpha\mu} \dot{x}^\mu \right] - \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \dot{x}^\mu \dot{x}^\nu = 0 \qquad ④$$

The method is: start from ④ and try to get a form like

$$\ddot{x}^\alpha + \text{terms} \propto \dot{x}\dot{x} = 0$$

Then from such an equation we can directly read the Christoffel Symbols.

We can get:

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{A'}{2A} \qquad \Gamma^r_{tt} = \frac{A}{2B} \qquad \Gamma^r_{rr} = \frac{B'}{2B} \qquad \Gamma^r_{\theta\theta} = -\frac{r}{B}$$

$$\Gamma^r_{\varphi\varphi} = -\frac{r\sin^2\theta}{B} \qquad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{A}{r} \qquad \Gamma^\theta_{\varphi\varphi} = -\sin\theta \cos\theta$$

$$\Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r} \qquad \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \cot\theta \qquad \text{and} \qquad A' \equiv \frac{dA}{dr} + \text{Remainders} = 0$$

## Riemann Tensor

In this case in unnecessary because of what happens to the Ricci tensor.
It can be directly computed from Christoffel Symbols

## Ricci Tensor

Remember that $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\alpha} + \Gamma^{\alpha}_{\alpha\rho}\Gamma^{\rho}_{\mu\nu} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\rho}_{\mu\alpha}$

So we can compute $R_{\mu\nu}$ directly from the metric. We obtain

$$R^t{}_t = \frac{A''}{2AB} - \frac{A'B'}{4AB^2} - \frac{A'^2}{4A^2B} - \frac{2B^2}{2rB^2}$$

$$R^r{}_r = \frac{A''}{2AB} - \frac{A'B'}{4AB^2} - \frac{A'^2}{4A^2B} - \frac{2B^2}{2rB^2}$$

$$R^\theta_\theta = R^\varphi_\varphi = \frac{A'}{2rAB} - \frac{B'}{2rB^2} - \frac{B-1}{r^2B}$$

## Solve $R^\mu{}_\nu = 0$

$R^t{}_t - R^r{}_r - R^\theta_\theta - R^\varphi_\varphi = 0$ is independent of $A, A', A''$.

All of the terms are 0 because of $R^\mu{}_\nu = 0$, the substraction will be 0 too then!

$$R^t{}_t - R^r{}_r - R^\theta_\theta - R^\varphi_\varphi = 0 \Rightarrow \frac{2B'}{rB^2} + \frac{2(B-1)}{r^2B} = 0$$

$$\frac{dB}{B(B-1)} + \frac{dr}{r} = 0 \Rightarrow \boxed{B(r) = \left(1 + \frac{a}{r}\right)^{-1}} \quad \text{where} \quad a = [L]$$

Now we consider another combination

$$R^t{}_t - R^r{}_r = 0 \Rightarrow \frac{A'}{rAB} + \frac{B'}{rB^2} = 0 \Rightarrow \frac{dA}{A} + \frac{dB}{B} = 0 \Rightarrow \boxed{A \cdot B = \text{constant}}$$

Physically this metric is a candidate to describe the space around a static and symmetrical object.

ANY MASSIVE STATIC OBJECT

The boundary condition will be:

$$\left.\begin{array}{l} A(r) \to c^2 \\ B(r) \to 1 \end{array} \right| \; \left. r \to \infty \right\} \text{ As } r \to \infty \implies g_{\mu\nu} \to \eta_{\mu\nu}$$

Finally

$$\boxed{A(r) = c^2\left(1 + \frac{a}{r}\right) \qquad B(r) = \frac{1}{1 + \frac{a}{r}}}$$

---

This should also reproduce Newtonian Limit

We can impose that it reduce to $g_{00} = -\left(1 + \frac{2\Phi}{c^2}\right)$ where $\Phi$ is the Newtonian Potential $\Phi = G_N M / r$

Putting everything toghether we find a:

$$\boxed{ds^2 = -\left(1 - 2\frac{G_N M}{c^2 r}\right)c^2 dt^2 + \left(1 - \frac{2 G_N M}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + \sin^2\theta \, d\varphi^2}$$

SCHWARZCHILD METRIC

$\rightsquigarrow$ For notation $r_S = \frac{2 G_N M}{c^2}$ is the Schwarzchild radius or S-radius

## BIRKOFF'S THEOREM

It state that spherical symmetric solutions of the vacuum EFE's is necessary static

Let's see the behaviour of the Sch-metric:

i) There is a singularity at $r = 0$

ii) Appears to be a singularity at $r = r_S$

iii) The t-like component of the metric becomes s-like when chosing $r_S$ value
$\rightsquigarrow$ This messes up with causalities $\implies$ We lose time variable

Let's see a couple of objects:

- Earth $r_s^S \simeq 1\,cm$ , $r_E \sim 6000\,Km$
- Sun $r_s^S \simeq 3\,Km$ , $r_S \sim 7\cdot 10^5\,Km$

} But is valid for EMPTY SPACE

When $r_s \sim r$ ? White black hole

---

## SCHWARZSCHILD COORDINATES INTERPRETATION

- $\theta, \varphi$: standard
- $t$ : Fair enough , it's essential time $-\infty < t < +\infty$
- $r$ : Sort of surface area radius.
  It has a geometrical interpretation

## Physics

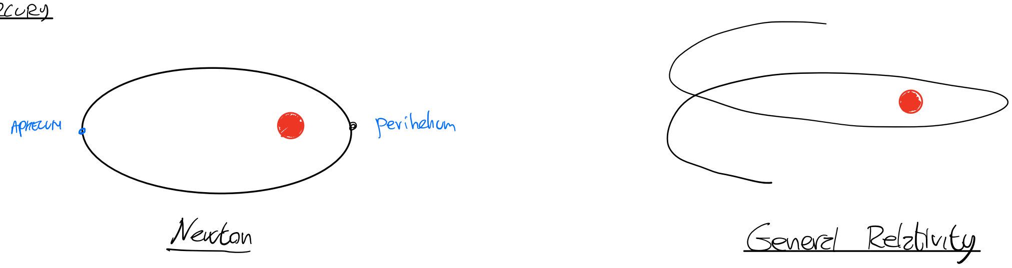
### SCHWARZSCHILD METRIC AT $r \gg r_s$

We could compute

i) Precession of perihelium of Mercury

ii) Light deflection by stars

### MERCURY




$\leadsto$ General Relativity predict no Newtonian Behaviour

Lagrangian

$$\mathcal{L} = g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}$$ we don't use $d\tau$ because is 0 for ple...

We saw that

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}\right) - \frac{\partial \mathcal{L}}{\partial x^\alpha} = 0$$

and that the S-metric is

$$ds^2 = -\left(1-\frac{r_S}{r}\right)c^2 dt^2 + \left(1-\frac{r_S}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$$\leadsto \mathcal{L} = -\left(1-\frac{r_S}{r}\right)c^2\dot{t}^2 + \left(1-\frac{r_S}{r}\right)^{-1}\dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right) = \begin{cases} -c^2 & \text{For massive objects} \\ 0 & \text{For photons} \end{cases}$$

Per simplicity we will restrict observers to EQUATORS $\Longrightarrow \theta = \frac{\pi}{2}$ and $\mathcal{L}$ simplifies to

$$\mathcal{L} = -\left(1-\frac{r_S}{r}\right)c^2\dot{t}^2 + \left(1-\frac{r_S}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\varphi}^2 = \begin{cases} c^2 \\ 0 \end{cases}$$

Note that

- $\mathcal{L}$ does not depend on $t$
- $\mathcal{L}$ does not depend on $\varphi$
- $\mathcal{L}$ does not depend on $\theta$

$\Big\}$ Cyclic COORDINATES

From Euler Lagrange Equations we get

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) = 0$$

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) = 0$$

$\Big\}$ These quantities are conserved in geodesics
Conserved quantities are energy and angular momentum

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = 0 \implies \text{Trivial}$$

From the NON-TRIVIAL EL equation consequence we get

$\left[ \left(1 - \frac{r_s}{r}\right)\dot{t} = \gamma \quad , \quad r\dot{\varphi} \right]$ where are constant

Lately we can always write $\quad \frac{dr}{d\lambda} = \frac{dr}{d\varphi}\frac{d\varphi}{d\lambda} \Rightarrow \dot{r} = \frac{dr}{d\varphi}\dot{\varphi}$

We saw

$$\mathcal{L} = -\left(1-\frac{r_S}{r}\right)c^2\dot{t}^2 + \left(1-\frac{r_S}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\varphi}^2 = \begin{cases} c^2 & \text{massive particles} \\ 0 & \text{photons} \end{cases} \qquad (A)$$

And conservation laws:

$$(B) \quad \left(1-\frac{r_S}{r}\right)\dot{t} = \gamma$$

$$(C) \quad r^2\dot{\varphi} = J$$

$$(D) \quad \dot{r} = \frac{dr}{d\varphi}\dot{\varphi}$$

$$(A),(B),(C),(D) \Rightarrow \left(\frac{dr}{d\varphi}\right)^2 = \frac{c^2\gamma^2 r^4}{J^2} - r^2 + r_S r \qquad \gamma = \left(1-\frac{r_S}{r}\right)\dot{t} \quad J = r^2\dot{\varphi}$$

DIFFERENTIAL EQUATION

To solve $\Rightarrow$ change of variable $u = \frac{1}{r}$

$$\Rightarrow \boxed{\left(\frac{du}{d\varphi}\right)^2 = r_S u^3 - u^2 + \beta}, \quad (E) \qquad \beta = \frac{c^2\gamma^2}{J^2}$$

We will get $r(\varphi)$ because we want to see how photons behaves in gravitational field

Formal Solution: $\varphi - \varphi_0 = \int_{u_0}^{u} \frac{du}{\sqrt{r_S u^3 - u^2 + \beta}}$ where $\varphi_0, u_0$: points on the trajectory

---

We know than

$$\frac{du}{d\varphi} = 0 \quad \text{at perihelion, aphelion}$$

---

So at PE / APH $\Rightarrow \frac{du}{d\varphi} = 0 \Rightarrow r_S u_\perp^3 - u_\perp^2 + \beta = 0 \Rightarrow \beta = u_\perp^2 - r_S u_\perp^3$

↳ When we hit PE/APH $u_\perp$ is a root of the polinomial

Now we can rewrite (E) as

$$\left(\frac{du}{d\varphi}\right)^2 = (u_\perp - u)\left[Au^2 + Bu + C\right]$$

A, B, C can be compute by matching the original polynomial

So doing that is

$$\left(\frac{du}{d\varphi}\right)^2 = \underbrace{(u_\perp - u)\left[(u_\perp+u) - r_s\left(u_\perp^2+u_\perp u+u^2\right)\right]}_{\text{Reproduce } \Rightarrow\ r_s u^3 - u^2 + \beta\text{ , but expanded}}$$

Replacing in our integral

$$\int_{u_0}^{u} \frac{1}{\sqrt{(u_\perp-u)\left[(u_\perp+u)-r_s\left(u_\perp^2+u_\perp u+u^2\right)\right]}}\,du$$

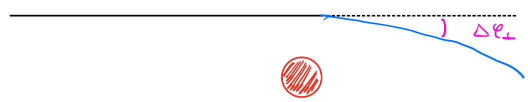
We are interested in $r \gg r_s$ , so we Taylor expand wrt $\varepsilon \equiv \frac{r_s}{r} = r_s u$

$$\varphi-\varphi_0 = \int_{u_0}^{u} \frac{du}{\sqrt{u_\perp^2-u^2}}\left[1+\frac{1}{2}r_s\left(\frac{u^2+uu_\perp+u_\perp^2}{u+u_\perp}\right)+\mathcal{O}(\varepsilon^2)\right]$$

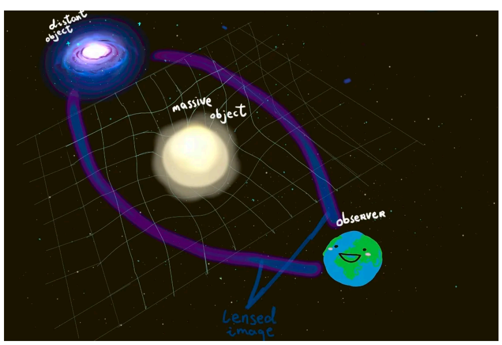
Integrate by switching to $q$ , defined through $u = u_\perp \frac{1-q^2}{1+q^2}$

As $-1 \leq q \leq +1$ , passing through $0$, $\Rightarrow$ $u$ starts from $0$ then come back to $0$, passing throw $u_\perp$

$$\Rightarrow \Delta\varphi = \underbrace{\pi}_{\text{Newton}} + \underbrace{2r_s u_\perp}_{\text{Leading Correction } \Delta\varphi_\perp}$$




Consequence: GRAVITATIONAL LENSING



S metric appear to have 2 singularities — $r=0$
— $r=r_s$

- $r=0$ is indeed singularity but not "naked"

At $r=r_s$, the $t$-variable becomes SPACE-LIKE

It's NOT A PHYSICAL TRUE SINGULARITY
IT'S A COORDINATE SINGULARITY

---

Consider radially ($\theta = \varphi = \text{const}$) photons

2 solutions

$$c^2\left(1-\frac{r_s}{r}\right)dt^2 - \frac{dr}{\left(1-\frac{r_s}{r}\right)} = 0 \quad \begin{cases} c\,dt = \dfrac{dr}{1-r_s/r} & \text{OUTGOING PHOTON} \\ -c\,dt = \dfrac{dr}{1-r_s/r} & \text{INCOMING PHOTON} \end{cases}$$

$$\Longrightarrow \begin{cases} ct = r + r_s \ln\left(\dfrac{r}{r_s}-1\right) + C & \text{OUTGOING} \\ -ct = r + r_s \ln\left(\dfrac{r}{r_s}-1\right) + D & \text{INCOMING} \end{cases}$$

Kruskal - Szekeres COORDINATES:

- $UV = e^{\frac{(C+D)}{2r_s}} = e^{\frac{r}{r_s}}\left(\frac{r}{r_s}-1\right)$
- $\pm\frac{V}{U} = e^{\frac{(C-D)}{2r_s}} = e^{\frac{ct}{r_s}} \longrightarrow +: r>r_s \quad -: r<r_s$

Let's see how metric transform if we perform this coordinate change

$$d(VU) = dVU + VdU = VU\left(\frac{dV}{V}+\frac{dU}{U}\right) = e^{\frac{r}{r_s}}\left[\frac{1}{r_s} + \frac{1}{r}\left(\frac{r}{r_s}-1\right)\right]dr$$

$$= \frac{VU}{\frac{r}{r_s}-1}\,\frac{dr}{\frac{r_s}{r}} = \frac{VU}{r_s\left(1-\frac{r_s}{r}\right)}$$

---

$$d\left(\frac{V}{U}\right) = \dots = \frac{V}{U}\left(\frac{dV}{U}-\frac{dU}{U}\right) = \dots = \left(\frac{V}{U}\right)\frac{c\,dt}{r_s}$$

Finally

$$\boxed{\frac{dV}{V}+\frac{dU}{V} = \frac{dr}{r_s\left(1-\frac{r_s}{r}\right)} \qquad \frac{dV}{V}-\frac{dU}{U} = \frac{c\,dt}{r_s}}$$

Then plug them back into the metric

$$ds^2 = \left(1 - \frac{r_s}{r}\right)\left[-c^2dt^2 + \frac{dr^2}{\left(1-\frac{r_s}{r}\right)^2}\right] = \left(1-\frac{r_s}{r}\right) r_s^2 \left[-\left(\frac{dV}{V} - \frac{dU}{U}\right)^3 + \left(\frac{dV}{V} + \frac{dU}{U}\right)^2\right] \Rightarrow$$

$$\boxed{ds^2 = -\frac{4r_s^3}{r} e^{-r/r_s} dV dU}$$

$$g_{\mu\nu} \to \begin{bmatrix} +\frac{4r_s^3}{r}e^{-\frac{r}{r_s}} & 0 & 0 & 0 \\ 0 & -\frac{4r_s^3}{r}e^{-\frac{r}{r_s}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2(\sin\theta)^2 \end{bmatrix}$$

No more singularity at $r = r_s$

Remark

- For given $U, V$ I can compute $r$

$$VU = \left(\frac{r}{r_s} - 1\right)e^{-\frac{r}{r_s}}$$

## Plot

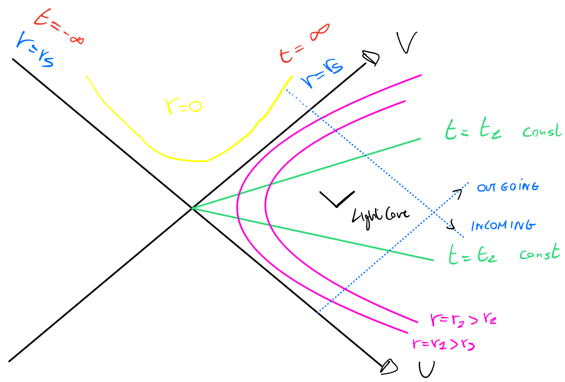
What do the Sch coordinates looks like in U,V plane?

1) $V=0$, $U=0 \Rightarrow VU=0$

$$\rightarrow e^{-r/r_s}\left(\frac{r}{r_s} - 1\right) = 0$$

$$r = r_s$$

→ Along both axes we are in $r = r_s$




2) $V=0 \Rightarrow \frac{V}{U} = 0 \Rightarrow e^{\frac{ct}{r_s}} = 0$

$$\rightarrow t = -\infty$$

$U=0 \Rightarrow \frac{V}{U} = \infty \Rightarrow e^{\frac{ct}{r_s}} = \infty$

$$\rightarrow t = \infty$$

3) $t = \text{const} \Rightarrow \frac{V}{U} = \text{const} \Rightarrow V = \text{const} \times U$ line const

4) $r = \text{const} \Rightarrow r > r_s \Rightarrow VU = \text{const}$ in $1^{st}$ quadrant

5) $r = 0 \Rightarrow VU = -1$

## Observer in Radial Free-Fall in Schwarzschild metric:

Lagrangian: $\mathcal{L} = c^2 \left(1 - \frac{r_s}{r}\right) \dot{t}^2 - \left(1 - \frac{r_s}{r}\right)^{-1} \dot{r}^2 = c^2$ ①

From previous problem: $\left(1 - \frac{r_s}{r}\right) \dot{t} = \gamma$ ② with $\gamma = \sqrt{1 + \frac{\dot{r}^2}{c^2} - \frac{r_s}{r}}$ $t$: cyclic

choose $\gamma = 1 \Rightarrow (\dot{r} = 0$ at $\infty)$

From ① and ②

$$\dot{r}^2 = \frac{r_s c^2}{r} \qquad \dot{t} = \frac{1}{1 - \frac{r_s}{r}}$$

Introduce $x$: $r = r_s x^2 \Rightarrow dr = 2 r_s x \, dx$

③ $\Rightarrow 2 r_s x^2 dx = c d\tau \Rightarrow c\tau = \frac{2}{3} r_s x^3$ (For $\gamma = 1$)

Assuming a static observer at $r > r_s$

④ $\Rightarrow dt = \frac{d\tau}{1 - \frac{r}{r_s}}$

Substituting in the results for $d\tau$, $r$ we obtain

$$c\,dt = \frac{2 r_s x^4 dx}{x^2 - 1} \Rightarrow ct = r_s \left[ \frac{2}{3} x^3 + 2x + \ln\left(\frac{x-1}{x+1}\right) \right]$$

As $x \to -1$ $\left(\to c\tau \to -\frac{2}{3} r_s, \; r \to r_s\right)$

# Cosmological Principle : Univers is Homogenous ⊕ Isotropic

We want to build an average Metric for the universe

1) <u>Copernican Principle</u>: The universe should look the same at any spatial point (Homogeneity)

2) <u>Isotropy</u>: The space looks no matter in what direction are you looking at.

## Friedmann - Lemaître - Robertson - Walker Metric (FLRW)

1) Isotropic Coordinates $(ct, x, y, z)$

<u>Impose That</u>

1) The symmetry has origin at $x = y = z = 0$

2) Line element of 3-space : $d\ell^2 = g\left(t, \underbrace{\sqrt{x^2+y^2+z^2}}_{\text{NOT THE PROPER DISTANCE FROM THE ORIGIN}}\right)$

Move to spherical coordinate $(t, r, \theta, \varphi)$

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \gg d\ell^2 = g(t,r)\left(dr^2 + r^2 d\theta + r^2\sin^2\theta\, d\varphi^2\right)$$

- Demand $g_{\mu\nu}$ invariant under $\begin{matrix}\theta \to -\theta \\ \varphi \to -\varphi\end{matrix} \implies g_{t\theta} = g_{t\varphi} = 0$

$$\rightsquigarrow ds^2 = f(t,r)\, c^2 d t^2 + g(t,r)\, dr^2 + h(t,r)\, dt\, dr + g(t,r)\, r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$f, g, h$ unknown functions

$\Longrightarrow$ $h(t,r)\, dt\, dr$ $\to$ ANNOYING but ERASABLE

Why? Consider example

$$ds^2 = -A(r)\, dt^2 + B(r)\, dr^2 + 2C(r)\, dr\, dt$$

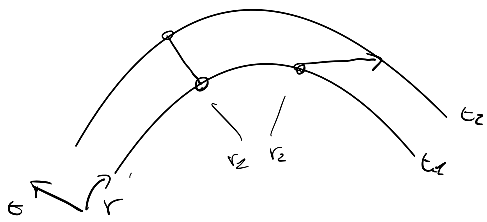
Introduce $T(t,r) \equiv t + \psi(r) \Rightarrow dT^2 = dt^2 + \psi'^2 dr^2 + 2\psi' dt\, dr \quad \left(\psi' \equiv \frac{d\psi}{dr}\right)$

$\to$ $C$ can be eliminated by choosing $\psi$ : $\frac{d\psi(r)}{dr} = -\frac{C}{A}$

$\rightsquigarrow$ Basically we can choose $C(r) = 0$

So as the example we can choose $h(r,t)=0$

Physical Explaination




2 observer at 2 different radii measures 2 different time $\Rightarrow$ AGAINST HOMOGENEOUS

Let's rewrite the metric with $h=0$:

$$ds^2 = -f(t,r)\,c^2dt^2 + g(t,r)\left[dr^2 + r^2d\theta^2 + r\sin^2\theta\, d\varphi^2\right]$$

OBSERVE THAT

Every static observer same $t$-measurement $\Rightarrow f = f(t) \Rightarrow$ does not depend on $r$

We can performe a coordinate transformation: $dt \to dt = \sqrt{F}\,dt$

"syncronisation"

OBSERVE THAT

From homogeneity: $g(t,r) = a^2(t)\, h(r)$

$$ds^2 = -c^2dt^2 + a^2(t)\, h(r)\left[dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\varphi^2\right]$$

We can perform coordinate transformation $\tilde{r}^2 = h(r)\, r^2$

$$\Rightarrow \quad ds^2 = -c^2dt^2 + a^2(t)\, W(\tilde{r})\, d\tilde{r} + a^2(t)\, \tilde{r}^2\left[d\theta^2 + \sin^2\theta\, d\varphi^2\right]$$

Spatial Part - Same generic form as S-metric

Recycle results from Schwarzchild ($\Lambda = 0$)

For spatial part

$$R_3 = \frac{2W'}{2rW^2} + \frac{W'}{rW^2} + \frac{2(W-1)}{r^2W}$$

Homogeneity $\Rightarrow R_3 =$ const in space $= 6K \qquad K \in \mathbb{R}$

$$W(\tilde{r}) = \frac{1}{1-Kr^2}$$

$$\boxed{ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta\, d\varphi^2\right]}$$

FLRW METRIC

## Comments

- $k = 0, \pm 1$
  - $0$ : closed, finite volume
  - $+1$ : flat, infinite volume
  - $-1$ : hyperboloid

- Ricci Scalar

$$R = 6\left[\frac{\ddot{a}}{a^2} + \frac{\dot{a}}{a^2} + \frac{k}{a^2}\right]$$

S-T flat if $k=0$ AND $a = \text{const}$

$a = 0$ is singularity

$a \to$ scale factor

- $k = +1$ : $r = \frac{1}{\sqrt{k}} \sin\chi$

$$d\ell^2 = d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

- $k = 0$ : $r \equiv \chi$

$$d\ell^2 = d\chi^2 + \chi^2(\dots)$$

- $k = -1$ : $r = \frac{1}{\sqrt{-k}} \sinh\chi$

$$d\ell^2 = d\chi^2 + \sinh^2\chi\,(\quad)$$

- It's possible to show that any world line of $\chi, \theta, \varphi = \text{const}$ is a geodesic and $t \sim \tau$

- For an observer at rest in the comoving frame:

  1) $\nexists$ local momentum flow

  2) Local stress : isotropic

- Observer at the coordinate origin — Observer at constant $\chi$

distance at: $\underline{a(t)\chi}$ distance depends on time

$\rightsquigarrow$ <u>Cosmological Redshift</u>

Fundamental Observer A at $x=0$

Light pulse emission at $t_A$

Fundamental observer at $x_B$ receives pulse at $t_B$

$$dt^2 - a^2 dx^2 = 0 \implies \int_{t_A}^{t_B} dt/a = x_B$$

At $t_A + \Delta t_A$, another pulse, received at $t_B + \Delta t_B$ | $x_A$ = chose $x_B$ = const

$$\Delta \int_{t_A}^{t_B} \frac{dt}{a(t)} = \frac{\Delta t_B}{a(t_B)} - \frac{\Delta t_A}{a(t_A)} = 0$$

$\Delta t_A$ : period of photon

Expansion :

# Gravitational Waves

## Electromagnetism Review

Lorentz: $m\ddot{\vec{x}} = e\vec{E} + \frac{e}{c}\dot{\vec{x}} \times \vec{B}$

Maxwell:

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$\vec{A}$: $\vec{B} = \vec{\nabla}\times\vec{A}$

$\phi$: $\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\nabla\varphi$

They can manifest from an action

$$S = S_n + S_{int} + S_{EM}$$

$S_{EM}$ → electromagn.

$S_{int}$ → Interaction Term

$S_n$ → Free Particle

$$S_m = -mc\int_{\Gamma}\sqrt{-ds^2}$$

$$S_{int} = \frac{e}{c}\int_{\Gamma} A_\mu\, dx^\mu = \frac{1}{c^2}\int_{\Omega} J^\mu A_\mu\, d^4\Omega$$

$$S_{EM} = -\frac{1}{16\pi c}\int_{\Omega} F^{\mu\nu}F_{\mu\nu}\, d^4\Omega$$

Defining:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$A^\mu = (\varphi, \vec{A})$$

$\varphi$ → scalar

$$J^\mu = (\rho c, \rho\vec{v})$$

$\rho$ → charge density

Inhomogeneous M-E: $\underbrace{\partial_\mu F^{\mu\nu} = -\frac{4\pi}{c} J^\nu}$

Invariant under $A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x)$

GAUGE INVARIANCE

$\Lambda$ is a generic function

Ex: Choose $A_\mu$ : $\partial_\mu A^\mu = 0$ "Lorentz Gauge" > Always Possible

Why all of this? In GR the same things happend because the Einst. Equation WORKS in EVERY COORDINATE SYSTEM

⟶ The physics remains unchanged

## Linearized Gravity

Same idea as Newtonian Limit

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

↳ Perturbation

Reminder: $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$

⇝ We will drop order of $h^2$, $\partial h^2$ Drop

Proof: $g^{\mu\nu} = \eta^{\mu\nu} + H^{\mu\nu}$

$$g_{\mu\nu} g^{\nu\rho} = (\eta_{\mu\nu} + h^{\mu\nu})(\eta^{\nu\rho} + H^{\nu\rho}) = \delta_\mu^{\ \rho} + \eta_{\mu\nu} H^{\nu\rho} + h_{\mu\nu}\eta^{\nu\rho} + \mathcal{O}(hH)$$

$$\longrightarrow H_\mu^{\ \rho} = -h_\mu^{\ \rho} = -h^{\mu\nu}$$

Remarks:

- Raise or lower indeces with $\eta^{\mu\nu} / \eta_{\mu\nu}$ (in linearised gravity)
- Summary:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$$

$$g^\mu_{\ \nu} = \delta^\mu_{\ \nu} \ (= \eta^\mu_{\ \nu}) \qquad \text{where} \quad h^\mu_{\ \nu} = \eta^{\mu\lambda} h_{\lambda\nu}$$

$$h^{\mu\nu} = \eta^{\nu\lambda} h^\mu_{\ \lambda}$$

so that $g_{\mu\nu} g^{\nu\rho} = \delta_\mu^{\ \rho}$

- $h_{\mu\nu}$ is TENSOR only under Minkowsky

# Einstein Field Equation in Linearized Gravity

## Christoffel Symbol

By definition $$\Gamma^{\lambda}_{\sigma\rho} = \frac{1}{2} g^{\lambda\mu} \left( \partial_\rho g_{\sigma\mu} + \partial_\sigma g_{\rho\mu} - \partial_\mu g_{\rho\sigma} \right)$$

- We require $O(h) \rightarrow g^{\lambda\mu} = \eta^{\lambda\mu}$

$$\rightsquigarrow \Gamma^{\mu}_{\sigma\rho} = \frac{1}{2} \eta^{\lambda\mu} \left( \partial_\rho h_{\sigma\mu} + \partial_\sigma h_{\rho\mu} - \partial_\mu h_{\rho\sigma} \right)$$

## Riemann Tensor

By definition $$R^{\lambda}{}_{\mu\sigma\nu} = \partial_\sigma \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\sigma} + \underbrace{\cancel{\Gamma\Gamma} - \cancel{\Gamma\Gamma}}_{O(h^2)}$$

$$\rightsquigarrow R^{\lambda}{}_{\mu\sigma\nu} \simeq \partial_\sigma \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\sigma}$$

$$\simeq \frac{1}{2} \eta^{\lambda\alpha} \cancel{\partial_\sigma ( \partial_\nu h_{\mu\alpha}} + \partial_\mu h_{\nu\alpha} - \partial_\alpha h_{\nu\mu} ) -$$

same

$$- \frac{1}{2} \eta^{\lambda\alpha} \cancel{\partial_\nu ( \partial_\sigma h_{\mu\alpha}} + \partial_\mu h_{\sigma\alpha} - \partial_\alpha h_{\sigma\mu} )$$

$\times \eta_{\rho\lambda}$

$$\rightsquigarrow R^{\lambda}{}_{\mu\sigma\nu} \simeq \frac{1}{2} \eta^{\lambda\alpha} \left( \partial_\sigma \partial_\mu h_{\nu\alpha} - \partial_\sigma \partial_\alpha h_{\nu\mu} - \partial_\nu \partial_\mu h_{\sigma\alpha} - \partial_\nu \partial_\alpha h_{\nu\mu} \right) \Rightarrow O(h)$$

$$\rightsquigarrow R_{\lambda\mu\sigma\nu} \simeq \frac{1}{2} \left( \partial_\sigma \partial_\mu h_{\nu\alpha} - \partial_\sigma \partial_\alpha h_{\nu\mu} - \partial_\nu \partial_\mu h_{\sigma\alpha} - \partial_\nu \partial_\alpha h_{\nu\mu} \right) \Rightarrow O(h)$$

Remark: • Those tensors satisfy Bianchi identities

## Ricci Tensor

By definition $$R_{\mu\nu} = g^{\lambda\sigma} \underbrace{R_{\lambda\mu\sigma\nu}}_{O(h)} \simeq \eta^{\lambda\sigma} R_{\lambda\mu\sigma\nu}$$

$= \eta^{\lambda\sigma}$

$$\Rightarrow R_{\mu\nu} = \frac{1}{2} \left( \partial_\mu \partial^\lambda h_{\nu\lambda} + \partial_\nu \partial^\sigma h_{\sigma\mu} - \Box h_{\mu\nu} - \partial_\mu \partial^\nu h \right)$$

$\partial^\lambda \downarrow \eta^{\sigma\lambda} \partial_\sigma$

$\rightarrow h = \eta^{\lambda\nu} h_{\mu\nu}$

$\rightarrow \Box = \eta^{\mu\nu} \partial_\mu \partial_\nu$

## Ricci Scalar

By definition $$R = g^{\mu\nu} R_{\mu\nu} \simeq \eta^{\mu\nu} R_{\mu\nu} = \frac{1}{2}\left(\partial^\nu\partial^\lambda h_{\nu\lambda} + \partial^\mu\partial^\sigma h_{\mu\sigma} - \Box h - \Box h\right)$$

$$\Longrightarrow R = \partial^\mu\partial^\nu h_{\mu\nu} - \Box h$$

## Einstein Equation

By definition $$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu}$$

$$\leadsto \frac{1}{2}\left(\partial_\mu\partial^\lambda h_{\nu\lambda} + \partial_\nu\partial^\sigma h_{\sigma\mu} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right) - \frac{1}{2}\eta_{\mu\nu}\left(\partial^\sigma\partial^\rho h_{\sigma\rho} - \Box h\right) = 8\pi G_N T_{\mu\nu}$$

**Trace Reversed perturbation**

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

Computing the TRACE $$\tilde{h} = \eta^{\mu\nu}\tilde{h}_{\mu\nu} = h - 2h \Longrightarrow \boxed{\tilde{h} = -h}$$

So we can invert $$h_{\mu\nu} = \tilde{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\tilde{h}$$

Substitute in Eistein Equation:

$$\frac{1}{2}\Big[\partial_\mu\partial^\lambda\tilde{h}_{\nu\lambda} \underset{-\frac{1}{2}A}{\cancel{- \frac{1}{2}\eta_{\nu\lambda}\partial_\mu\partial^\lambda\tilde{h}}} + \partial_\nu\partial^\sigma\tilde{h}_{\sigma\mu} \underset{-\frac{1}{2}A}{\cancel{- \frac{1}{2}\eta_{\sigma\mu}\partial_\nu\partial^\sigma\tilde{h}}} - \Box\tilde{h}_{\mu\nu} + \cancel{\frac{1}{2}\eta_{\mu\nu}\Box\tilde{h}} \underset{+A}{\cancel{+ \partial_\mu\partial_\nu\tilde{h}}}\Big] - \frac{1}{2}\eta_{\mu\nu}\Big(\partial^\sigma\partial^\rho\tilde{h}_{\sigma\rho} - \underbrace{\cancel{\frac{1}{2}\eta_{\sigma\rho}\partial^\sigma\partial^\rho}\tilde{h}}_{\Box} + \Box\tilde{h}\Big) = 8\pi G_N T_{\mu\nu}$$

$\Longrightarrow$ Einstein Eqs:

$$\boxed{\partial_\mu\partial^\lambda\tilde{h}_{\nu\lambda} + \partial_\nu\partial^\sigma\tilde{h}_{\sigma\mu} - \Box\tilde{h}_{\mu\nu} - \eta_{\mu\nu}\partial^\sigma\partial^\rho\tilde{h}_{\sigma\rho} = 16\pi G_N T_{\mu\nu}}$$

## Harmonic Gauge

Intrestingly, linearized gravity does by no means, uniquely specify the perturbation $h_{\mu\nu}$.

⇝ The freedom to perform an infinitesimal coordinate transformation $x^\mu \to x'^\mu = x^\mu + \xi(x^\mu)$ give rise to AN INVARIANCE UNDER GAUGE TRANSFORMATION

$\underbrace{\partial_\mu \tilde{h}^{\mu\nu} = 0}_{\text{Choice}}$ Harmonic / Hilbert / De Donder gauge

→ I go in the coordinate in wich the $\tilde{h}^{\mu\nu}$ has THAT FORM

We get

$$\boxed{\Box \tilde{h}_{\mu\nu} = -16\pi G_N T_{\mu\nu}}$$

> Thus, $\bar{h}_{ab}$ represents a quantity that propagates as a wave at the vacuum speed of light, on a flat Minkowski background, and which is sourced by the energy-momentum tensor $T_{ab}$ of matter; in other words, $\bar{h}_{ab}$ is a *gravitational wave.*

$\Rightarrow \Box \xi_\nu = \partial^\mu \tilde{h}_{\mu\nu}$ → Brings us to harmonic gauge

N.B. If we are in the armonic gauge Ⓐ with $\xi : \Box\xi = 0 \Rightarrow$ We stay in harmon. Gauge

→ The formal solution of E. Eq

$$\tilde{h}_{\mu\nu}(t, \vec{x}) = 4 G_N \int_{\substack{\text{over}\\ \text{flat}\\ \text{space}}} d^3\vec{x}' \, \frac{T_{\mu\nu}(t - |\vec{x} - \vec{x}'|, \vec{x}')}{|\vec{x} - \vec{x}'|}$$

### Transverse - Tracless (TT) gauge

$\tilde{h}_{\mu\nu}$: A prior: 10 indipendent component

$\tilde{h}_{\mu\nu}$ → symmetric

Harmonic Gauge condition ⇒ $\tilde{h}_{\mu\nu}$ → 6 ind. components

Which means thet the harmonic gauge condition does not uniquely specify the metric perturbation $h_{\mu\nu}$.

INDEED any gauge transformation whose generator $\xi^a$ satisfeys $\Box \xi_a = 0$ ⇝ Does preserve the GAUGE CONDITION

To uniquely specify the metric perturbation, 4 additional CONSTRAIN must be IMPOSED

$\Rightarrow \tilde{h} \to 2$ indipendent (phisical components)

In practice

1) Choose $\xi^0$ : $\hat{h} = 0$ (i.e. $\hat{h}_{\mu\nu} \to$ traceless)

2) Choose $\xi^i$'s : $h_{0i} = 0 \Rightarrow \partial^\mu \hat{h}_{\mu\nu} = 0 \Rightarrow \partial^\mu h_{\mu\nu} = 0 \Rightarrow \partial^0 h_{00} = 0$

$\Rightarrow \underbrace{h_{00}}$ is $t$ - independent

$\to$ Newtonian independent

$$\Rightarrow \boxed{h_{0\mu} = 0 \qquad h = 0 \qquad \partial^i h_{ij} = 0}$$

Notation: We write e.g. $\tilde{h}^{TT}_{\mu\nu}$

N.B. In vacuum, $\Box \hat{h}_{\mu\nu} = 0$ or choose $\Box \xi_\mu = 0$ $\to$ TT gauge always possible in vacuum

$\neq$

With matter $\Box \hat{h}_{\mu\nu} \neq 0$

$\to$ Not always possible to go to TT gauge

## Gravitational Waves : Propagation in vacuum

$\Box \hat{h}^{TT}_{\mu\nu} = 0$ $\rightarrow$ It's a <u>Wave equation</u>

Assuming $h^{TT}_{0\mu} = 0$ $\Rightarrow$ Plane wave solution as

$$h^{TT}_{bc} = \varepsilon_{bc}\, e^{ik^{\mu}x_{\mu}}$$

where

- $K^{\mu} = (\omega, \vec{K})$, $\omega = |\vec{K}|$ (angular frequency) of Gr. Wav.
- $\varepsilon_{bc}$ = pol. tens
- $\omega^2 = |\vec{K}|^2 \rightarrow$ Disp. Relation

For propagation in $z$-direction

$$h^{TT}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cos[\omega(t - z/c)]$$

Where $h_+, h_\times$ : amplitudes of polarizations models

We can also write:

$$h^{TT}_{bc} = h_+ \varepsilon^{+}_{bc} \cos[\omega(t-z)] + h_\times \varepsilon^{\times}_{bc} \cos[\omega(t-z)]$$

using

$$\varepsilon^{+}_{bc} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\varepsilon^{\times}_{bc} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then the LINE ELEMENT

$$\boxed{ds^2 = -c^2 dt^2 + (1 + h_+ \cos\varphi)\, dx^2 + (1 - h_+ \cos\varphi)\, dy^2 + 2 h_\times \cos\varphi\, dx\, dy + dz^2}$$

$\rightsquigarrow \varphi \equiv \omega(t - z/c)$

## 1) Single particle in path of GR

Free Particle : $\ddot{x}^{\mu} + \Gamma^{\mu}_{\nu\rho} \frac{dx^{\mu}}{d\tau} \frac{dx^{\rho}}{d\tau} = 0$

Particle at rest at $\tau = 0$ $\rightarrow$ At $\tau = 0$ : $\frac{dx^{i}}{d\tau} = 0$

$$\Rightarrow \ddot{x}^{\mu} + \Gamma^{\mu}_{00} = 0 \quad \Rightarrow \ddot{x}^{\mu} = -\Gamma^{\mu}_{00} = \frac{1}{2} \eta^{\mu\lambda} \left( \partial_0 h_{0\lambda} + \partial_0 h_{0\lambda} - \partial_\lambda h_{00} \right)$$

In TT-gauge : $h_{0\lambda} = 0 \Rightarrow \ddot{x}^{\mu} = 0$ (at $\tau = 0$) $\Rightarrow$ Particle at Rest

But Rest with respect to his coordinate system

What if we consider the proper distance between 2 particles?

## 2) Two particles at $(x_0, 0, 0)$ , $(-x, 0, 0)$

How does the proper distance evolve?

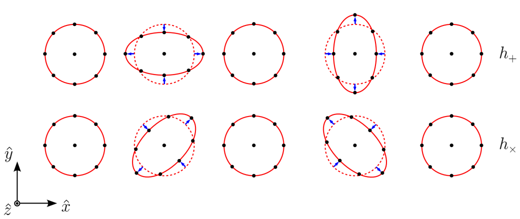
$$\Delta l = \int_{-x_0}^{x_0} (|ds|)^{1/2} = \int_{-x_0}^{x_0} \sqrt{|g_{xx}\, dx^x dx^x|}$$

$$= \int_{-x_0}^{x_0} \sqrt{1 + h_{xx}}\, dx' \underset{\text{Taylor}}{\simeq} L_0 \left[ 1 + \frac{1}{2} h_+ \cos(\omega t) \right] \qquad L_0 = 2x_0$$

N.B. $(0, x_0, 0)$ , $(0, -x_0, 0)$ $\Rightarrow$ $\Delta l = L_0 \left[ 1 - \frac{1}{2} h_+ \cos(\omega t) \right]$

Comments :

1) Proper distance do change
2) $\Delta l \propto L_0$ (initial separation) $\Longrightarrow$ GW Detectors are large
3) If you rotate the system by 45° $\Rightarrow$ Same answer but $h_\times$
4)



## Eistein's Quadrupole Formula

We shall describe the generation of gravitational waves by isolated systems.
For each component of $\tilde{h}_{\mu\nu}$ of the perturbation, the linear equation can be solved as:

$$\tilde{h}_{\mu\nu}(t,\vec{x}) = 4G_N \int d^3\vec{x}\, \frac{T_{\mu\nu}(t-|\vec{x}-\vec{x}'|, \vec{x}')}{|\vec{x}-\vec{x}'|} \qquad \text{in harmonic gauge}$$

- Assume compact source: $|\vec{x} \to \vec{x}'| \simeq |\vec{x}| \equiv r$

$$\Rightarrow \tilde{h}_{\mu\nu}(t,r) = \frac{4G_N}{r} \int d^3\vec{x}'\, T^{\mu\nu}(t-r, \vec{x}')$$

- In linearized theory $h_{\mu\nu}$ and $T_{\mu\nu} \to$ Same order (Must Be). This is telling us that

$$\Rightarrow D_\mu T^{\mu\nu} = 0 \quad \to \partial_\mu T^{\mu\nu} = 0$$

Starting from this, we can play around with components

1. $\partial_\mu T^{\mu\nu} = 0 \Rightarrow \partial_0 T^{0\nu} = -\partial_k T^{k\nu}$

Integrate over region with $T^{\mu\nu} \neq 0$

$$\frac{\partial}{\partial t}\int_V T^{0\nu} d^3\vec{x} = -\int_V \frac{\partial T^{k\nu}}{\partial x^k} d^3\vec{x} \underset{\text{Gauss}}{=} -\int_\Sigma T^{k\nu} d\Sigma_k = 0$$

$$\Rightarrow \int_V T^{0\nu} d^3\vec{x} = \text{const} \Rightarrow \tilde{h}^{0\nu} = \text{const} \to \text{Let's decide to put } \tilde{h}^{0\nu} = 0$$

2. $\partial_\mu T^{\mu i} = 0 \Rightarrow \partial_0 T^{0i} = -\partial_k T^{ki} \Rightarrow x^j \partial_0 T^{0i} = -x^j \partial_k T^{ki}$

Integrate

$$\frac{\partial}{\partial t}\int x^j \partial_0 T^{0i} = -\int x^j \partial_k T^{0i} \overset{\text{by part}}{=} -\int_V \frac{\partial}{\partial x^k}(x^j T^{ik})\, d^3\vec{x} + \int_V T^{ik}\frac{\partial x^j}{\partial x^k}\, d^3\vec{x} =$$

$$= -\int_\Sigma T^{ik} \underbrace{\cancel{x^j}}_{=0} d\Sigma_k + \int_V T^{ij} d^3\vec{x} = \int_V T^{ij} d^3\vec{x}$$

S.En.Ten vanish

But $T^{\mu\nu}$ is symmetric $\Rightarrow$ Same result with $i \leftrightarrow j$

Add the two $\Rightarrow \frac{\partial}{\partial t}\int_V (T^{i0}x^j + T^{j0}x^i)\, d\vec{x} = 2\int_V T^{ij} d^3\vec{x}$ (1)

3) $\partial_\mu T^{\mu 0} = 0 \implies \partial_0 T^{00} = -\partial_i T^{i0} \implies x^i x^j \partial_0 T^{00} = x^i x^j \cdot (-\partial_k T^{k0})$

$$\implies \int_V x^i x^j \partial_0 T^{00} \, d^3\vec{x} = -\int_V x^i x^j \partial_k T^{k0} \, d^3\vec{x} + \int_V \left( T^{k0} \frac{\partial x^i}{\partial x^k} x^j + T^{k0} x^i \frac{\partial x^j}{\partial x^k} \right) d^3\vec{x}$$

$$= \int x^i x^j T^{k0} + \int_V \left( T^{j0} x^j + T^{i0} x^j \right) d^3\vec{x} \implies$$

$$\implies \frac{\partial}{\partial t} \int_V x^i x^j \partial_0 T^{00} d^3\vec{x} = \int_V \left( T^{j0} x^j + T^{i0} x^j \right) d^3\vec{x}$$

$$\implies \frac{\partial^2}{\partial t^2} \int_V x^i x^j \partial_0 T^{00} d^3\vec{x} = \frac{\partial}{\partial t} \int_V \left( T^{j0} x^i + T^{i0} x^j \right) d^3\vec{x} \implies$$

$$= 2 \int_V T^{ij} \, d^3\vec{x} \qquad \text{using } ①$$

Definition: Quadruple moment of source

$$\ddot{Q}^{ij}(t) = \int_V T^{00}(t, \vec{x}) \, x^i x^j \, d^3\vec{x} \quad \rightsquigarrow \quad \ddot{Q} \equiv \frac{\partial^2 Q}{\partial t^2}$$

Solution for EFE:

$$\tilde{h}_{\mu 0} = 0$$

$$\tilde{h}_{ij} = \frac{2G_N}{r} \ddot{Q}_{ij}(t-r)$$

$\rightsquigarrow$ if I have sferic symmetry g. waves collapse

Move to tt gauge $\to$ Preserve harmonic
$\to$ Switches to TT gauge

- Already $\tilde{h}_{\mu 0} = 0$
- Trackless condition $\delta^{ij} h_{ij}^{TT} = 0$
- Transverse condition $n^i h_{ij}^{TT} = 0$ where $\hat{n} = \frac{\vec{x}}{r}$

In order to do that we will introduce some operators

$$\boxed{P_{ij} \equiv \delta_{ij} - n_i n_j}$$ → symmetric
→ Projects vectors on the plane transverse to $\vec{n}$

Btw → $P_{ij} n^i = \delta_{ij} n_i - n_i n_j n^i = n_j - n_j = 0$

- Introduce $\boxed{P_{ijkl} \equiv P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl}}$

$$h_{ij}^{TT} = P_{ijkl} \tilde{h}_{kl}$$

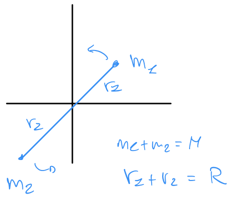
Exercise Prove that $h_{ij}^{TT}$ is traceless $(\delta^{ij} h_{ij}^{TT} = 0)$

- In the end :

$$\begin{cases} h_{\mu 0}^{TT} = 0 \\ h_{ij}^{TT} = \frac{2G_N}{r} Q_{ij}^{TT}(t-r) \end{cases}$$

Where $Q_{ij}^{TT} = P_{ijkl} Q_{kl}$

## Binary (s) in orbital orbit.




$$\begin{cases} m_1 r_1 + m_2 r_2 = 0 \\ r_1 = \frac{m_2 R}{M} \qquad r_2 = \frac{m_2 R}{M} \end{cases}$$

1) Consider newtonian orbit

$$\frac{G_N m_1 m_2}{R^2} = m_1 \Omega^2 r_1 = \frac{m_2 m_2 R}{M}\Omega^2 \Longrightarrow \Omega^2 = \sqrt{\frac{G_N M}{R^3}}$$

2) Trajectories

$$\vec{x}_1 = \begin{pmatrix} r_1 \cos \Omega t \\ r_1 \sin \Omega t \\ 0 \end{pmatrix} \qquad \vec{x}_2 = \begin{pmatrix} -r_2 \cos \Omega t \\ -r_2 \cos \Omega t \\ 0 \end{pmatrix}$$

- $T^{00} = \sum_{i=1}^{2} m_i c^2 \, \delta(x-x_i)\, \delta(y-y_i)\, \delta(z)$

$$\Longrightarrow Q^{ij}(t) = \int_V T^{00}(t,\vec{x})\, x^i x^j \, d^3\vec{x}$$

So $$Q_{xx} = \sum_{i=1}^{2} \int m_i c^2 \, \delta(x-x_i)\delta(y-y_i)\delta(z)\, x^2 \, dx\, dy\, dz$$

$$= m_1 x_1^2(t) + m_2 x_2^2(t) = \frac{\mu R^2}{2}\cos \Omega t + \text{const}$$

where $\mu = \frac{m_1 m_2}{m_1+m_2}$

In the end

$$Q_{ij} = \frac{\mu R^2}{2}\begin{pmatrix} \cos(2\Omega t) & \sin(2\Omega t) & 0 \\ \sin(2\Omega t) & -\cos(2\Omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} (+\text{const})$$

- Assuming observer along $z$-axis
- Unit vector $(0,0,1)$
- Projection $P_{ij} = \delta_{ij} - n_i n_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$Q_{xx}^{TT} = P_{xxij} Q^{ij} = \left( P_{xi} P_{xj} - \frac{1}{2} P_{xx} P_{ij} \right) Q^{ij}$$

$$= \left( P_{xx}^2 - \frac{1}{2} P_{xx}^2 \right) Q_{xx} - \frac{1}{2} P_{xx} P_{xy} Q_{xy} - \frac{1}{2} (Q_{xx} - Q_{yy})$$

Then $$Q_{ij}^{TT} = \begin{pmatrix} \frac{1}{2}(Q_{xx} - Q_{yy}) & Q_{xy} & 0 \\ Q_{xy} & -\frac{1}{2}(Q_{xx} - Q_{yy}) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So $$h_{\mu 0}^{TT} = 0$$

$$h_{ij}^{TT} = \frac{2G_N}{r} \ddot{Q}_{ij}^{TT}(t-r) \Rightarrow h_{\mu\nu}^{TT} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} h_+(t) &= -\frac{4 G_N \mu R^2 \Omega^2}{c^4 z} \cos\left[ 2\Omega \left( t - \frac{z}{c} \right) \right] \\ h_\times(t) &= -\frac{4 G_N \mu R^2 \Omega^2}{c^4 z} \sin\left[ 2\Omega \left( t - \frac{z}{c} \right) \right] \end{aligned}}$$

<u>Comparison with EM</u>