Quantum Electrodynamics as a Gauge Theory

Quantum Electro Dynamics as a Gauge Theory

I. Introduction

0: Electromagnetic and Relativity Recap

Maxwell's equation

$$\bullet\ \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \bullet\ \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial}{\partial t}\vec{E} = \mu_0\vec{j}$$

$$\bullet\ \vec{\nabla}\cdot\vec{B} = 0 \qquad \bullet\ \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial}{\partial t}\vec{B} = 0$$

They are COVARIANT under Lorentz Transformation: $x^\mu \to x'^{\mu'}(x) = \Lambda^{\mu'}{}_{\mu}\, x^\mu$ (The form stay the same from one inertial reference frame to another)

Which is manifested by defining $x^\mu = (x^0, \vec{x}) = (ct, x, y, z)$ dimension of space

LET'S FIND THE COVARIANT FORM OF THESE EQUATIONS:

But you can define also a current

$$J^\mu \to (c\rho, \vec{j})$$

And you need to define

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ +E_x/c & 0 & -B_y & +B_y \\ & & 0 & -B_x \\ & & & 0 \end{pmatrix} = -F^{\nu\mu}$$

Antisymmetric

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ \cdot & 0 & E_z/c & -E_y/c \\ \vdots & & 0 & E_x/c \\ & & & 0 \end{pmatrix} = -\tilde{F}^{\nu\mu} = \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

$\varepsilon^{0123} = +1 = -\varepsilon^{0213}$ (totally antisymmetric)

$\Rightarrow$ Then Maxwell becomes

$$\begin{cases} \partial_\mu F^{\mu\nu} = \mu_0 J^\nu & (1) \\ \partial_\mu \tilde{F}^{\mu\nu} = 0 & (2) \end{cases}$$

Covariant form of Maxwell equations

Now looking at $F^{\mu\nu}$ ($\Leftrightarrow \vec{B}, \vec{E}$) 6 compononents.

Of 6 compononents, 4 of them are constrained by (2)

The general solution of (2) is $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ for some vector field $A^\mu(x)$

We know $A^\mu = (A^0, \vec{A}) = \left(\frac{\varphi}{c}, A_x, A_y, A_z\right) = \left(\frac{\varphi}{c}, A^x, A^y, A^z\right)$

However there is an invariance

$A'^{\mu} = A^{\mu} + \partial^{\mu}\theta(x)$ gives the same $F^{\mu\nu}$ as A^{μ}

$\rightarrow \quad F'^{\mu\nu} = \partial^{\mu}A'^{\nu} - \partial^{\nu}A'^{\mu} = F^{\mu\nu} + \underbrace{\cancel{\partial^{\mu}\partial^{\nu}\theta - \partial^{\nu}\partial^{\mu}\theta}}_{=0} = F^{\mu\nu}$

e.g. $\partial^1\partial^2\theta$
$= (-\partial_1)(-\partial_2)\theta = \partial_2\partial_1\theta = \partial^2\partial^1\theta$
(partial derivatives commute...)

$\rightarrow$ Gauge Invariance : $\neq A^{\mu}$ give the same physical F

Furthermore: J^{ν} (which creates $E, B \neq 0 \Rightarrow A \neq 0$) only has 3 independent components since

$\partial_{\nu} J^{\nu} = \frac{1}{\mu_0}\, \partial_{\nu}\partial_{\mu} F^{\mu\nu} \equiv 0$ current must be conserved

Aim of the lecture:

Since only 2 components of A^{μ} are physical,

HOW TO PICK THEM IN A LORENTZ INVARIANT WAY $\rightsquigarrow$ (Also at quantum level)

II. Relativistic Spin 1 particles

a) Lagrangian

At rest, we want 3 different components/states related by ROTATION.

In Non Relativistic Quantum Mechanics to describe a SPIN 1 particle we use
$j = 1, \; m = -1, 0, 1$

We want to describe a particle : WE NEED A FIELD.

- If you have 3 different components, you need 3 FIELD COMPONENTS

$\Longrightarrow$ WE REQUIRE TO DESCRIBE $S=1$ particle $\rightarrow$
- 4-vector Field $A^\mu(x)$ (Need 4 to ensure Lorentz-Invariance)
- \+ Lorentz-invariant way to kill 4^{TH}-component
- \+ Free particles/field of mass $m \neq 0$ obey K-G like equation

FIRST TRY

Klein-Gordon Lagrangian of A^μ field (For the spacial part)

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{3}(\partial_\mu A^i)^2 - \frac{m^2}{2}\sum_i\left(A^i(x)\right)^2$$

(e.g. $A^1 = A^2 \simeq \varphi$ scalar field)

$\rightarrow$ Rotational Invariance

$\rightarrow$ NOT Lorentz invariance

$$-\frac{1}{2}(\partial_\mu A^0)^2 + \frac{m^2}{2}A^{0\,2}$$

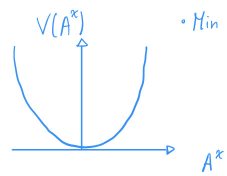
Now, try to write $\mathcal{L}$ in a COVARIANT WAY

$$= -\frac{1}{2}(\partial_\mu A^\alpha)(\partial^\mu A^\beta)\eta_{\alpha\beta} + \frac{m^2}{2}A^\alpha A^\beta \eta_{\alpha\beta}$$

$$\overset{!}{=} \frac{1}{2}A^\alpha \eta_{\alpha\beta}(\Box + m^2)A^\beta$$

$i=1$

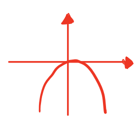
$$\underbrace{\frac{1}{2}(\partial_0 A^x)^2}_{E_K \geq 0} \underbrace{-\frac{1}{2}(\partial_x A^x)^2 \underbrace{- \frac{m^2}{2}(A^x)^2}_{\text{Potential}}}_{-(E_p>0)}$$



• Min E_p is $A^x = 0$

$\rightarrow$ With KG we have stable at $t=0$ (because of kinetic energy)
and we have $A^x(t,\vec{x}) = 0$ BUT

$\rightarrow$ WRONG SIGN! $\not\!\!A$ energy



At quantum level it is even worse

$$A^x \to a^{x\dagger}(\vec{k}) \quad \text{with} \quad [a^x, a^{x\dagger}] = +1 \quad (\times\ \delta(k-k)\ldots)$$

so that $a^{x\dagger}|0\rangle$ is a positive norm state:

$$\underbrace{\langle 0| a^x a^{x\dagger} |0\rangle}_{=+1} = \langle 0|0\rangle \times 1 + \langle 0| \underbrace{a^{x\dagger} a^x |0\rangle}_{=0}$$

Lorentz Generalization

$$A^\mu \to a^{\mu\dagger}(\vec{k}) \quad \text{with} \quad [a^\mu, a^{\nu\dagger}] = -\eta^{\mu\nu}$$

$$\to [a^0, a^{0\dagger}] = -1$$

$\to a^{0\dagger}|0\rangle$ is a NEGATIVE NORM STATE = GHOST

$$\Rightarrow \langle 0| a^0 \; a^{0\dagger} |0\rangle = -1 \; !$$

Second Try

Proca's trick to get rid of A^0 in a covariant way

$$\mathcal{L}_{proca} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{m^2}{2}(A_\mu)^2 - J^\nu A_\nu$$

$$= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)\eta^{\mu\mu'}\eta^{\nu\nu'}(\partial_{\mu'}A_{\nu'} - \partial_{\nu'}A_{\mu'}) + \frac{m^2}{2}A_\mu A^\mu - J^\nu A_\nu$$

$$= \underbrace{-\frac{1}{4}(\partial_\mu A_\nu)^2(1+1)}_{\text{Like First Try}} + \underbrace{\frac{1}{4}(\partial_\mu A_\nu)(\partial^\nu A^\mu)(1+1)}_{\text{New Term}} + \frac{m^2}{2}A_\mu A^\mu - J^\nu A_\nu$$

So the action

$$\delta S_{proca} = \oint d^4x \ \frac{2}{2}(\delta^\nu A_\nu)\left[\partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu + m^2 A^\nu - J^\nu\right] = 0 \quad \forall \delta A_\nu(x)$$

$$\boxed{(\Box + m^2)A^\nu - \partial^\nu \partial_\mu A^\mu = 0} \quad \text{Proca Equation}$$

$$= J^\nu$$

b) Solutions

$$\partial_\nu\left(\begin{matrix}\text{Proca}\\ \text{Equat}\end{matrix}\right) = (\Box + m^2 - \Box)\,\partial_\mu A^\mu = 0 \quad = \partial_\nu J^\nu \quad \Rightarrow \quad \boxed{\partial_\mu A^\mu = \frac{1}{m^2}\partial_\nu J^\nu}$$

if $J^\nu = 0 \ \Rightarrow \ \partial_\mu A^\mu = 0$

CASE $J^\nu = 0$.

The solutions are PLANE WAVE

Makes sense to separate our field into spacetime and internal part: $A_\mu(x_\mu) = \varepsilon_\mu \times e^{-iKx}$

with constrain, obtained by plug the WAVE into Proca's equation

(2) $\boxed{K^2 = K^{0\,2} - \vec{K}^2 = m^2} \Rightarrow K^0 = \pm\omega(\vec{K}) = \pm\sqrt{\vec{K}^2 + m^2}$

(1) $\boxed{\varepsilon \cdot K = 0} \Rightarrow \varepsilon^0 = 0$ if $\vec{K} = 0$ at rest $K^\mu = (m, \vec{0})$

$\Rightarrow$ So we have 3 independent polarizations $\varepsilon^\mu_{(\lambda)}$ with $\boxed{\varepsilon_{(\lambda)} \cdot \varepsilon_{(\lambda')} = -\delta_{\lambda\lambda'}}$

- At rest $K = (m, 0, 0, 0)$

$$\varepsilon^\mu_{(1)} = (0, 1, 0, 0) \qquad \varepsilon^\mu_{(2)} = (0, 0, 1, 0) \qquad \varepsilon^\mu_{(3)} = (0, 0, 0, 1)$$

NOT COMPLETELY DONE IN CLASS

This implies that for a field excitation that travels in the z direction there are in total 3 linear independent polarization structures

It's convenient to choose specific basic blocks to describe all possible polarizations

like LAST PAGE

$$\boxed{\text{We could trivially take } i=1,\dots,4 \text{ and use } \varepsilon^{i}_{\mu}(p)=\delta^{i}_{\mu}\text{. Instead we want a basis that forces } A_\mu(x) \text{ to automatically satisfy its equation of motion } \partial_\mu A^\mu = 0.}$$

$\Rightarrow$ This will happend if $\boxed{K_\mu \varepsilon^{\mu}_{i}(x)=0}$

For any 4-momentum K^μ with $\boxed{K^2=m^2}$ there are 3 independent solutions to this equation given by 3 4-vectors $\varepsilon^{i}_{\mu}(x)$ $\Rightarrow$ POLARIZATION VECTORS

- We conventionally normalize the polarizations by $\boxed{\varepsilon^*_\mu \varepsilon_\mu = -1}$

Let's be explicit by choosing a canonical basis.

Take K^μ to point in the z direction.

$$K^\mu = (E, 0, 0, K_z) \quad \rightarrow \quad E^2 - K_z^2 = m^2 \quad \checkmark$$

Then the 2 obvious vectors satisfying $K_\mu \varepsilon_\mu = 0$ and $\varepsilon^*_\mu \varepsilon_\mu = -1$ are:

$$\left.\varepsilon^1_\mu = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad \varepsilon^2_\mu = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\} \text{TRANSVERSE POLARIZATION}$$

So the other one is $\Rightarrow$

$$\left.\varepsilon^L_\mu = \begin{pmatrix}K_z/m\\0\\0\\\frac{\sqrt{m^2+K_z^2}}{m}\end{pmatrix}\right\} \text{LONGITUDINAL POLARIZATION}$$

The third is linear independent

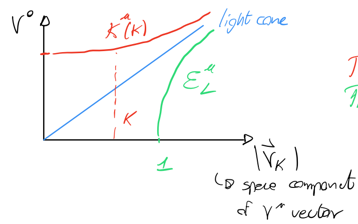
$\rightarrow$ m in ε^L_μ is a normalization factor $\left(\varepsilon^L_\mu \varepsilon^{L\mu} = \frac{K_z^2}{m^2} - \frac{\left(\sqrt{m^2+K_z^2}\right)^2}{m^2} = -1 \right)$

Comments:

1| For the photon, that is massless, we do not have $\mathcal{E}_L$ ~ MASSLESS GAUGE (different condition)
The $\mathcal{E}$ of γ are ORTHOGONAL to the direction of motion

2| If $\vec{k} \to \infty$ or $\vec{k} \gg m$ we will have

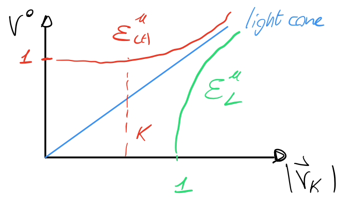
$$\mathcal{E}_L = \left(\frac{k}{m} , \frac{\omega}{m} \frac{\vec{k}}{k} \right) \longrightarrow \text{the components goes to } \infty \,!$$



Taking the v^μ 4-vector
The RED is the K^μ 4-vector
The GREEN is the $\mathcal{E}_L^\mu(k)$

• In the $\vec{k} \to \infty$ $\mathcal{E}_L^\mu$ and K^μ are indistinguishable

Since $K^\mu = m\, \mathcal{E}^\mu_{(t)}$



There are also CIRCULAR POLARIZATIONS:

$$\boxed{\mathcal{E}^\mu_{(\pm)} = \frac{1}{\sqrt{2}} \left(\mathcal{E}^\mu_{(1)} \pm i\, \mathcal{E}^\mu_{(2)} \right) = \mathcal{E}^{\mu\,*}_{(\mp)}} \in \mathbb{C}$$

so you can define the third component $\mathcal{E}^\mu_{(0)} = \mathcal{E}^\mu_L \in \mathbb{R}$

$$\Rightarrow \boxed{\mathcal{E}^*_{(\lambda)} \cdot \mathcal{E}_{(\lambda)} = -\delta_{\lambda \lambda'}} \quad \text{where} \quad \lambda, \lambda' = (-1, 0, +1) \quad \text{helicity}$$

These precise types of polarization vectors are EIGENVECTORS of helicity operators

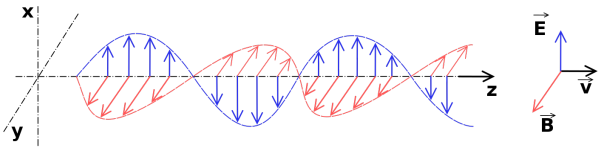
$$\underbrace{\left(\frac{\vec{K}}{K} \cdot \vec{S} \right)}_{\text{HELICITY OPERATOR}} \mathcal{E}_{(\lambda)} = \lambda\, \mathcal{E}_{(\lambda)}$$

Now we want to rewrite $A^\mu(x) = \mathcal{E}^\mu(x)\, e^{-ikx}$ in this formalism.

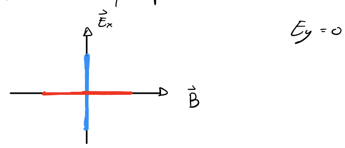
NOT DONE IN CLASS

Yes! But let's try to get an intuitive understanding of this POLARIZATION.

We know from Electromagnetism that the light is an alternate oscillation of the electric and magnetic field:



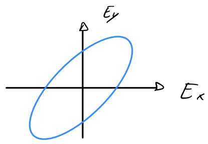
Visualizing this on $\vec{B}, \vec{E}$ plane is



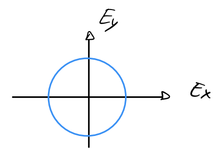
This is called LINEAR POLARIZATION

But could happen that the direction of the Electric Field is not linear, but could be elliptical or CIRCULAR.

ELLIPTICAL:



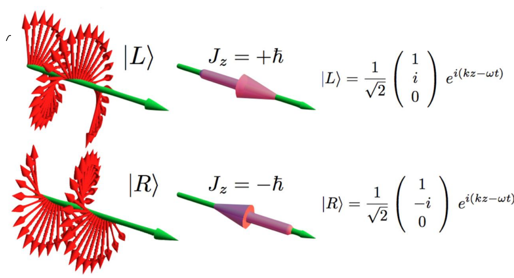
CIRCULAR:



Looking at the circular polarization, we now know that it means that the electric field is SPINNING around the direction of motion.

But the spinning could happen clock wise ~ RIGHT or ANTI clock wise ~ LEFT

When a light beam is CIRCULARLY POLARIZED each of its photons carries a spin angular momentum of $\pm\hbar$.
+ and − stand for RIGHT and LEFT circular polarization



So that's why we talked about polarization.

This concept could be extended to any field.

c) General solution of Proca equation with $\varepsilon_{(\lambda)}$ polarization vectors

$$A^{\mu}(x) = \int \frac{d^4 x^{\mu}}{(2\pi)^2} \tilde{A}^{\mu}(K)\, e^{-ikx}$$

$$\tilde{A}^{\mu}(X) = \int \frac{d^4 X}{(2\pi)^2} A^{\mu}(x)\, e^{ikx}$$

so $\underbrace{\tilde{A}^{\mu}(K)^{*} = \tilde{A}^{\mu}(-K)}_{\text{save it for later}}$

For $J^{\mu} = 0$ then

$$\boxed{\tilde{A}(k) = \hat{A}_{+}(\vec{k})\, \varepsilon_{+}^{\mu}(\vec{k})^{*} + \hat{A}(\vec{k})_{-}\, \varepsilon_{-}^{\mu}(\vec{k})^{*} + \hat{A}_{(0)}(k)\, \varepsilon_{0}^{\mu}(\vec{k})}$$

IN CIRCULAR

will become annihilation operator

with

If you look at your component for the polarization

$$\tilde{A}_{(\lambda)}(\vec{k}) = -\,\varepsilon_{(\lambda)}^{\mu}(\vec{k}) \cdot \tilde{A}_{\mu}(k)$$

, can extract by taking the scalar product

e.g. For $\vec{k} = (0, 0, K)$

$$\tilde{A}_{\pm}(\vec{k}) = \frac{1}{\sqrt{2}} \left(\underbrace{\tilde{A}^{x}(\vec{k})}_{x} \pm i\, \underbrace{\tilde{A}^{y}(K)}_{y} \right)$$

$$\tilde{A}_{0}(\vec{k}) = \frac{1}{m} \left(\underbrace{\hat{A}^{z}(\vec{k})}_{z}\, \omega(\vec{k}) - \underbrace{\hat{A}^{t}(k)}_{t} K \right)$$

2.1 Feynmann Rules : Propagators

$$S[A^\nu(x)] = \int d^4x \left(-\frac{1}{2}\right) \partial_\mu A_\nu(x)\left[\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)\right] + \frac{m^2}{2} A_\nu(x) A^\nu(x) - J^\nu(x) A_\nu(x)$$

$$= \int d^4x \; -\frac{1}{2}\int \frac{d^4K}{(2\pi)^2}\left(-iK_\mu \tilde{A}_\nu\right)\tilde{A}_\nu e^{-ikx}\left[\int \frac{d^4K'}{(2\pi)^2} e^{-ik'x}\left[-iK'^\mu \tilde{A}^\nu + iK'^\nu \tilde{A}^\mu\right]\right] +$$

$$\int d^4x \int \frac{d^4K}{(2\pi)^2}\int \frac{d^4K'}{(2\pi)^2} \tilde{A}_\nu(K)\left(\tilde{A}^\nu(K')\frac{m^2}{2} - \tilde{J}^\nu(K')\right)e^{-i(K+K')\cdot x}$$

Now in both double integrals, let's use the Fourier Transform of the Dirac's Delta δ

$$\boxed{\int d^4x \; e^{-i(K+K')x} = (2\pi)^4 \delta^{(4)}(K+K')} \Rightarrow K' = -K$$

Then
1. The delta is killing the integral over K'
2. $(2\pi)^N$ simplify

$$= \int d^4K \left[\frac{m^2}{2}\tilde{A}_\nu(K)\overbrace{\tilde{A}^\nu(K)^*}^{\tilde{A}^\nu(-K)} - \tilde{A}_\nu(K)\overbrace{\tilde{J}^\nu(K)^*}^{\tilde{J}^\nu(-K)} - \frac{1}{2}\left(K_\mu K^\mu \tilde{A}_\nu \overbrace{\tilde{A}^{\nu *}}^{\tilde{A}^\nu(-K)} - K^\nu \tilde{A}_\nu K_\mu \overbrace{\tilde{A}^{\mu *}}^{\tilde{A}^\mu(-K)}\right)\right]$$

$$= \int d^4K \; \tilde{A}^\mu(K)^* \left[\underbrace{\frac{1}{2}\left(-K^2 + m^2\right)\delta^\nu_\mu + K_\mu K^\nu}_{M^\nu_\mu(K)}\right]\tilde{A}_\nu(K) - \tilde{J}^\nu(K)^* \tilde{A}_\nu(K)$$

Now let's make a variation of $S \to \delta S = 0$ $\gg$ We can compute the equation of motion for A^μ

We get (simplifing $\tilde{A}_\nu(K)$ ...

$$\boxed{M^\nu_\mu \tilde{A}^{\mu *} = \tilde{J}^{\nu *}}$$

To solve this equation let's remove the stars, and solve it

$$M^\nu_\mu(K)\tilde{A}^\mu(K) = \tilde{J}^\nu(K)$$ $\to$ This has to be true for all the K

So a solution is formally $\tilde{A}^{\mu}(K) = \underbrace{(M^{-1})^{\mu}{}_{\nu}}_{\text{as something to do with a propagator}} J^{\nu}(K)$

The PROPAGATOR is $G^{\mu}{}_{\nu}(K)$

$$iG^{\mu}{}_{\nu}(K) = i(M^{-1})^{\mu}{}_{\nu}$$

Example

IF $M(K) = K^2 - m^2 \Rightarrow$ propagator $= \dfrac{i}{K^2 - m^2}$

Let's try with our propagator

$$G^{\mu}{}_{\nu} = \underbrace{F(K^2)\delta^{\mu}{}_{\nu}}_{\text{This is covariant}} + B(K^2)K^{\mu}K_{\nu}$$

$\Rightarrow$ we are imposing this shape

why? Lorentz Covariance is required

$$G^{\mu}{}_{\nu}M^{\nu}{}_{\rho} = \left(F\delta^{\mu}{}_{\nu} + B(K^2)K^{\mu}K_{\nu}\right)\left(-(K^2-m^2)\delta^{\nu}{}_{\rho} + K^{\nu}K_{\rho}\right)$$

$$= \underbrace{-F\cdot(K^2-m^2)}_{=1}\delta^{\mu}{}_{\rho} - K^{\mu}K_{\rho}\underbrace{\left[\cancel{+B(K^2-m^2)} - F - \cancel{BK^2}\right]}_{=0}$$

Needs to $\hookrightarrow = \delta^{\mu}{}_{\rho}$ IF $G = M^{-1} \Rightarrow \boxed{(M^{-1})^{\mu}{}_{\nu}\cdot M^{\nu}{}_{\rho} = \delta^{\mu}{}_{\rho}}$

$$\Rightarrow \begin{cases} F = \dfrac{-1}{+K^2 - m^2} \\ B = \dfrac{-F}{m^2} \end{cases} \Rightarrow \boxed{iG = \frac{-1i}{K^2-m^2}\left[\delta^{\mu}{}_{\nu} - \frac{K^{\mu}K_{\nu}}{m^2}\right]} \quad (*)$$

Feynmann Propagator

Or easily by using QFT Feynmann Trick $\quad iG = \dfrac{-i}{K^2-m^2}\left[\delta^{\mu}{}_{\nu} - \dfrac{K^{\mu}K_{\nu}}{m^2}\right]$

NOT DONE IN CLASS

WHY WE COMPUTED THE PROPAGATOR LIKE THIS?

⇝ In QFT we computed the Feynman Propagator:

$$G(x,y) = \langle 0 | T[\varphi(x), \varphi(y)] | 0 \rangle$$

by doing all the computation, solving the integral in the complex $\mathbb{C}$ space we obtained

$$G(x,y) = \lim_{\varepsilon \to 0} \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{i}{p^2 - m^2 + i\varepsilon}$$

Then we discussed about the consequence

Consequence a) In the momentum space $\tilde{G}(p) = \dfrac{i}{p^2 - m^2 + i\varepsilon}$

Consequence b) QFT 12

- If you take $\underbrace{(\Box_x + m^2)}_{\text{Equation of motion}} G_F(x,y) = -i\,\delta^4(x-y)$

⟶ Go to momentum space

⟹ So we have basically done the same : equation of motion
Green Function
Find the inverse ⟹ it's the propagator

Riemann Lemma

What if NO POLES?

$$f(t) = \int_{-\infty}^{+\infty} d\omega \, e^{-i\omega t} \hat{f}(\omega)$$

If $\hat{f}(\omega)$ is smooth $\rightarrow$ $f(\pm\infty) = 0$ (so the oscillation will average to 0)

To NOT HAVE 0 at ∞ we need POLE $\Big\} \longrightarrow \dfrac{1}{\omega - \omega_0 + i\varepsilon} \xrightarrow[T\to\infty]{} e^{-i\omega_0 t}$

$\hookrightarrow$ This is the reason why we need POLES for an infinite propagation DOESN'T BLOW UP

$\Rightarrow$ Whatever we do to the harmonic oscillator it will propagate to infinity BECAUSE THERE IS A POLE

Let's look at the propagator itself

Notice:

(1) $G^{\mu}{}_{\nu} \equiv 4\times 4$ matrix, like Dirac Propagator

So we have 4 propagating degrees of freedom? NO!

IF $J^{\mu}(K) = K^{\mu} J(K^2)$

$$\Rightarrow \tilde{A}^{\mu}(K) = G^{\mu}{}_{\nu}(K)\, J^{\nu}(K) = \frac{-1}{K^2 - m^2}\left(\frac{m^2 - K^2}{m^2}\right) K^{\mu} J(K^2) = \frac{1}{m^2} J(K^2)$$

NO POLE AT $K^2 = m^2$!

$\Rightarrow A^{\mu}(x) = \dfrac{1}{m^2} J^{\mu}(x)$ does not propagate to ∞ if J^{μ} is located in Space Time

This comes from the fact that

(2) $P^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - \dfrac{K^{\mu} K_{\nu}}{m^2} = (P^2)^{\mu}{}_{\nu}$ if $K^2 = m^2$

$\hookrightarrow$ Projector of vectors $\perp K^{\mu}$ $\gg$ $P^{\mu}{}_{\nu} K^{\nu} = K^{\mu} - \dfrac{K^2}{m^2} K^{\mu} = 0$ if $K^2 = m^2$

3. On shell we can write $P^{\mu}{}_{\nu}$ in the $(*)$ as

$$\boxed{\underline{P^{\mu}{}_{\nu}(K)} = \sum_{\lambda=0,\pm1} -\varepsilon^{\mu}_{(\lambda)}(\vec{K})\,\varepsilon^{*}_{(\lambda)\nu}(\vec{K}) = \left(P^{\mu}{}_{\nu}\right)^{*}}$$

It's a projector, in QM $P = \sum_n |n\rangle\langle n|$ with $|n\rangle$ base vectors

$\rightsquigarrow$ The -1 may seem strange, but $\varepsilon^{*}_{(\lambda)}\,\varepsilon_{(\lambda')} = -\delta_{\lambda\lambda'}$

2.2 Feynmann Rules: Wave Functions and external legs.

Those rules are about amplitude and polarization and external legs.

The general solutions of the equation of motion $\underline{\text{for } J^\mu = 0}$ NO SOURCE

$$A^\mu(x) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}\sqrt{2W(\vec{k})}} \sum_{\lambda=0,\pm1} a_\lambda(\vec{k})\, \underbrace{\varepsilon^\mu_\lambda(\vec{k})}_{\text{Normalized}}\, e^{-ikx} + \underbrace{a^+_\lambda(\vec{k})\,\varepsilon^\mu_\lambda(\vec{k})}\, e^{ikx}$$

NOTICE: $\sum_\lambda a^+_\lambda(\vec{k})\,\varepsilon^\mu_\lambda(\vec{k})^* = -P^\mu_\nu\, a^+(\vec{k})$

w/ $\varepsilon_{(\lambda)} \cdot \varepsilon_{(\lambda')} = -\delta_{\lambda\lambda'}$

$a_\lambda(\vec{k})$ → Coefficient of the plane wave solution, normalized LIKE K-G case

Side-note:

<u>When we quantize</u> the system, we

$a^+_\lambda(\vec{k}) = -\varepsilon_{(\lambda)}(\vec{k})^\nu\, a^+_\nu(\vec{k})$ → we will obtain that $a^+_\lambda(\vec{k})$ will become the CREATION OPERATOR of ON-SHELL PARTICLES with helicity λ and momentum $\vec{k}$ (λ is not summed)

like Klein-Gord

This then will lead to

$$[a^+_{(\lambda)}(\vec{k}),\, a_{(\lambda')}(\vec{k}')] = \delta^{(3)}(\vec{k}-\vec{k}')\,\delta_{\lambda\lambda'}$$

(each of coeff is like the Klein Gordon)

Each of this coefficients are like a KG-Field. Notice that we do

$$\sum_\lambda a^+_\lambda(\vec{k})\,\varepsilon^\mu_\lambda(\vec{k}) = -P^\mu_\nu(\vec{k})\, a^{\nu+}(\vec{k})$$

Then to derive the other Feynmann rules we have to start from the S matrix.

The invariant S MATRIX elements / amplitude between initial (n-j) and final state j

$$M_{\underbrace{\lambda_1,\dots,\lambda_j}_{\text{FINAL } j},\underbrace{\lambda_{j+1},\dots\lambda_n}_{\text{INITIAL } (n-j)}} = \underbrace{\langle \vec{k}_1\lambda_1, \vec{k}_2\lambda_2, \dots, \vec{k}_j\lambda_j|}_{\langle 0|a_{\lambda_1}(\vec{k}_1)\dots a_{\lambda_j}(\vec{k}_j)|} S \underbrace{|\vec{k}_{j+1}\lambda_{j+1}, \dots, \vec{k}_n\lambda_n\rangle}_{|a^+_{\lambda_{j+1}}(\vec{k}_{j+1})\dots a^+_{\lambda_n}(\vec{k}_n)|0\rangle}$$

Helicities

For each one there are 3 possibilities for the helicity (could happen that $h_i = 0, 1, -1$)

So it's clear that the matrix element depends on the helicity λ of the particles. Sure.

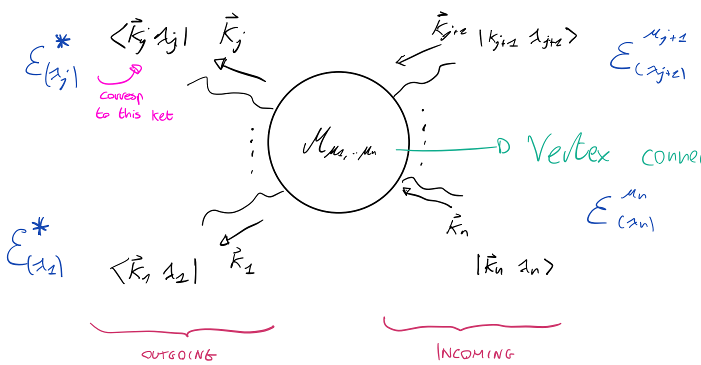
Now, the MATRIX ELEMENT MUST BE INVARIANT, and in some ways this is also invariant, but does not appear to be that because of the Helicity $\Rightarrow$ Seems that we have JUST 3 degrees of Freedom, where is the Fourth one?
How do they MIX when we make a Lorentz transformation?

To MAKE IT MORE APPEARING, the trick is to write down the matrix element with LORENTZ INDIECES

$$\mathcal{M}_{\lambda_1, \dots, \lambda_j, \lambda_{j+1}, \dots \lambda_n} = \underbrace{\mathcal{M}_{\mu_1, \dots, \mu_n}}_{\text{lorentz 4-index}} \left(\mathcal{E}^{\mu_1 *}_{(\lambda_1)} \dots \mathcal{E}^{\mu_j *}_{(\lambda_j)} \right) \left(\mathcal{E}^{\mu_{j+1}}_{(\lambda_{j+1})} \dots \mathcal{E}^{\mu_n}_{(\lambda_n)} \right)$$

$\Rightarrow$ Manifestly Lorentz Invariant
BUT WE HAVE STILL TO DEFINE $\mathcal{E}^{\mu}_{(\lambda)}$ IN AN INVARIANT WAY. NOT EASY see next.

Let's make a PICTURE



$\Rightarrow$ Vertex connecting n A_μ fields.
Only makes sense for $n \geq 3$
$\rightsquigarrow$ interaction

Now we know that the PROBABILITY $\propto |\mathcal{M}_{\lambda_1 \dots}|^2$

$\longrightarrow$ IF we want the UN-POLARIZED PROBABILITY we need to average the OUTGOINGS

$$\text{Un-P} = \underset{\lambda_{IN}}{\text{Average}} \sum_{\lambda_{OUT}} |\mathcal{M}_{\lambda_1 \dots}|^2$$

So

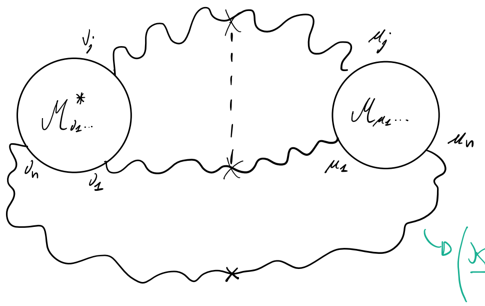
$$dP \propto \frac{1}{3^{(n-j)}} \sum_{\lambda_1, \lambda_2, \dots, \lambda_n} |\mathcal{M}_{\lambda_1, \lambda_2, \dots}|^2 = \frac{1}{3^{(n-j)}} \sum_{\lambda_2, \dots \lambda_n} \left(\underline{\underline{\mathcal{E}^{\mu_1}_{\lambda_1}}} \dots \mathcal{M}_{\mu_1 \dots} \right) \left(\underline{\underline{\mathcal{E}^{\nu_1 *}_{\lambda_1}}} \dots \mathcal{M}^*_{\nu_1 \dots} \right)$$

$$= \underbrace{\left(\frac{k_1^{\mu_1} k_1^{\nu_1}}{m^2} - \eta^{\mu_1 \nu_1} \right)}_{-\text{Residue at the pole of } G^{\mu_1 \nu_1}} \times (1 \to 2) \times (1 \to n) \cdots \mathcal{M}_{\mu_1} \dots \mathcal{M}^*_{\nu_1} \dots$$

WHY?

$$\sum_{r=1}^{3} \epsilon^r_\mu \epsilon^r_\nu = -\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k_\rho k^\rho} \right).$$

Explanation Graph:



$$\to \left(\frac{K_n^{\mu_n} K_n^{\nu_n}}{m^2} - \eta^{\mu_n \nu_n} \right)$$

Projector

Summary

$$-i \left(\eta^{\mu\nu} - \frac{K^\mu K^\nu}{m^2} \right) \cdot \frac{1}{K^2 - m^2 + i\varepsilon}$$

$|\vec{K}, \lambda\rangle$ $\varepsilon^\mu_{(\lambda)}$ INCOMING

$\langle \vec{K}, \lambda|$ $\varepsilon^{\mu *}_\lambda$ OUTGOING

Vertices? Later.

⊕ Manifestly Lorentz covariant : $\varepsilon^\mu_{(\lambda)}$ are 4-vectors!

⊕ Only 3 propagating fields : Even if it's manifestly covariant, we have 3 degrees of freedom (Comes from Proca Trick)

⊖ Bad high energy behaviour. Why?

$\to$ $\varepsilon^\mu_{(0)}$ longitudinal polarization is singular when $|\vec{K}| \to \infty$

$$= \left(\frac{|\vec{K}|}{m}, + \frac{\omega}{m} \frac{\vec{K}}{K} \right) \xrightarrow[K \to \infty]{} \underbrace{\left(\frac{\omega}{m}, \frac{\vec{K}}{m} \right)}_{\varepsilon^\mu_+} = \frac{K^\mu}{m} \to \infty$$

$\to$ Propagator for $K^2 \to \pm\infty$ will be $\sim + \frac{i}{m^2} \frac{K^\mu K_\nu}{K^2} \to$ constant!
(like internal line non on-shell)

NOT WHAT'S IN K-G
(in K-G propagator $\to 0$)

Will lead to Non-renormalizable : BAD!

You can actually use what we have done till now to describe mass bosons $W^\pm_i$ Z, and you have a description that works, but if you measure just LONGITUDINAL W for example you understand that this theory does NOT WORK. TO GIVE THEM MASS YOU NEED HIGGS.

III. Spin 1 $m=0$ particles

{ Since we want to do QED, we'd WANT to study the case with $m=0$. Can we study the $m \to 0$ limit of what we've just done? }

Simple answer: For $m=0$, just only for $\varepsilon^{\mu}_{(0)}$ is very tricky!

Another way to see the deep difference $\leadsto$ When we FIX K^{μ} (time like vector), WHAT ARE THE TRANSFORMATIONS WHICH LEAVE THE K^{μ} INVARIANT?

NOT DONE IN CLASS

Reminder: Definitions

TIME-LIKE K^{μ} IF $K_{\mu}K^{\mu} > 0 \Rightarrow$ Associated with PARTICLE w/ MASS
$(K^{\mu}K_{\mu} = m^2)$ $\leadsto$ In the Rest-Frame of the particle K^{μ} is TIME-LIKE

LIGHT-LIKE K^{μ} = $K_{\mu}K^{\mu} = 0 \Rightarrow$ Associated with MASSLESS PARTICLE
$(K^{\mu}K_{\mu} = 0)$
$\leadsto$ There is no rest frame, with $\vec{k} = 0 \ldots$

Explain: why we want the K^{μ} remains invariant?

[Remember when we derived A_{μ} like wave with constrains? One constrain was $K^{\mu}\varepsilon_{\mu} = 0$
$\to$ The requirement of Lorentz invariance ensures that the set of polarizations vectors form a complete basis for the transverse spatial directions.

- Completeness? Means that any transverse vector can be expressed as a linear combination of the POLARIZATION VECTORS

Now, since $K^{\mu}\varepsilon_{\mu} = 0$, we know that under a Lorentz transformation, the polarization vectors remain in the PLANE perpendicular to the direction of motion (invariant plane), so the INVARIANCE of K^{μ} ensure that the polarization vectors are transformed in a way that preserve orthogonality to K^{μ}
$\to$ This guarantees that the physical interpretation of polarization states remains consistent across different reference frames

] not so clear...

TIME LIKE K^μ $\to$ I CAN GO IN THE REST FRAME $\Rightarrow$ K^μ will be $K^\mu = (k^0, 0,0,0)$ $\Rightarrow$
- cannot do Boost
- CAN DO rotations (invariant under $SO(3)$)

$\Rightarrow$ Works Fine

LIGHT LIKE K^μ $\to$ (There is NO REST FRAME) $\Rightarrow$ Whatever RF, cannot have spatial components equal 0 that will solve the problem. (like above)

Spatial tranformation that leaves $\vec{k}$ invariant? Only $SO(2)$ rotation around $\vec{k}$!

Groups are called Little Groups, and they are $\neq$!!!

So we discovered that there is a fundamental difference in fixing LIGHT-LIKE or TIME LIKE K^μ.

CONSEQUENCE:

LIGHT-LIKE $\Rightarrow$ $\nexists$ an $\perp$-normalized basis which contains $\varepsilon^\mu \propto K^\mu$ $(\underbrace{\varepsilon \cdot \varepsilon \propto K \cdot K = 0}_{\text{I cannot use } K^\mu \text{ as a basis vector!}})$!

So we are missing something

Let's try to use $\varepsilon^{(1)}, \varepsilon^{(2)}$.

$\varepsilon_{(1)}, \varepsilon_{(2)}$ we know that are TRANSVERSE to K^μ $\Rightarrow$ $\varepsilon \cdot K = 0$ as before and those will become our TRANSVERSE POLARIZATION.

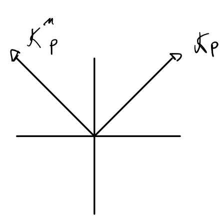
With them is easy $V^\mu = \varepsilon^\mu_{(1)} V^1 + \varepsilon^\mu_{(2)} V^2 \Rightarrow V^\mu = \eta^{\lambda\mu} \varepsilon_{(\lambda)} V$ $\mu = 1,2$ AS USUAL, VECTOR WRITTEN IN COMPONENTS USING THE BASIS VECTOR

But now, if I want a full basis, I need 2 other vectors!

Let's try with K^μ as first but then I need a vector, which is LINEAR INDEPENDENT with $\varepsilon^{(1)}, \varepsilon^{(2)}$ AND K^μ.

It's K_p^μ $\leadsto$ $K_p^\mu = (K^0, -\vec{K})$ $\leadsto$ PARITY FLIPPED of $K^\mu = (K^0, \vec{K})$

- If K^μ is light-like, so will K_p^μ ✓
- $\varepsilon_{(1,2)} K = \varepsilon_{(1,2)} K_p = 0$ ✓



✓

- $K \cdot K_P = (K_0)^2 + \vec{K}^2 = 2|\vec{K}|^2 \rightsquigarrow K^2 = K_P^2 = 0 \rightsquigarrow K \cdot K_P \neq 0$

So a general vector in this basis will be

$$V^\mu = \underbrace{\varepsilon^\mu_{(1)} V^1 + \varepsilon^\mu_{(2)} V^2}_{\text{TRANSVERSE}} + K^\mu V^3 + K^\mu_P V^4$$

Let's now take the scalar product:

d

$$V \cdot V = -(V^1)^2 - (V^2)^2 + V^3 V^4 \cdot 4|\vec{K}|^2$$

We can write the METRIC $\Rightarrow$ $(V^1 V^2 V^3 V^4) \left(\begin{array}{cc|cc} -1 & & & \\ & -1 & & \\ \hline & & 0 & 2|\vec{K}|^2 \\ & & 2|\vec{K}|^2 & 0 \end{array} \right) \begin{pmatrix} V^1 \\ V^2 \\ V^3 \\ V^4 \end{pmatrix}$

Tricky thing: EXTRACT V^3 and V^4, we need a scalar product

$$V^3 = \frac{1}{2|\vec{K}|^2} (K_P \cdot V) \qquad V^4 = \frac{1}{2|\vec{K}|^2} (K \cdot V)$$

coeff. of K

scalar product with K_P!!!

Important consequence

$$K^\mu + K^\mu_P = (2K^0, 0, 0, 0) \Rightarrow (K^\mu_P)' \neq (K'^\mu)_{P'}$$

Lorentz Transform.

$\rightarrow$ it's NOT THE SAME before and after the transformation

$\Rightarrow$ When you fix a Light-Like vector YOU GET MORE CONSTRAINS respect to fixing a time-like.

Maybe because in $m \neq 0$ there are 3 polariz., in $m = 0$ just 2, so more constrains.

So basically we can take $m \to 0$ but we have to change the basis (+1 constrain)

3.1. Lagrangian, Equation of Motion

Can we follow the Proca's Lagrangian method we used before?

For $m=0$ $\mathcal{L}_{Proca}$ and the Equations of Motions are:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - J^{\mu}A_{\mu} \qquad \text{with} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\frac{\delta S}{\delta A_\nu(x)} = 0 \longrightarrow \partial_\mu F^{\mu\nu} = \Box A^\nu - \partial^\mu(\partial_\mu A^\mu) = J^\nu(x)$$

$$\overset{\text{Fourier Transform}}{\Longleftarrow} \quad \boxed{\underbrace{(-K^2\delta^\nu_\mu + K^\nu K_\mu)}_{M^\nu_\mu(K)} \tilde{A}^\mu(K) = \tilde{J}^\nu(K)} \quad (5)$$

$$\Rightarrow \tilde{A}^\mu(K) = (M^{-1})^\mu_\nu \tilde{J}^\nu(K)$$

Doesn't work because M^ν_μ has 0-eigenvalue, not only for $K^2=0$, but for any K^μ

$$M^\nu_\mu K^\mu = (-K^2\delta^\nu_\mu + K^\nu K_\mu)K^\mu = -K^2K^\nu + K^\nu K^2 \equiv 0$$

In the case of MASSIVE PARTICLES there was the mass term that prevented the $=0$.

More precisely since every vector field can be split into a LONGITUDINAL and TRANSVERS part

$$M^\nu_\mu = (-K^2)\, P_{T\,\mu}^{\ \nu}(K)$$

with

$$P_{T\,\mu}^{\ \nu} = \delta^\nu_\mu - \frac{K^\nu K_\mu}{K^2} = (P_T^2)^\nu_\mu = \text{Projection on transverse vector space}$$

$$P_{L\,\mu}^{\ \nu} = \frac{K^\nu K_\mu}{K^2} = (P_L^2)^\nu_\mu = \text{Longitudinal projector on } K^\mu$$

$\Rightarrow (M^{-1})$ can only be defined in the transverse subspace and the longitudinal part $\tilde{A}^\mu \sim K^\mu$ cannot be fixed by (5)

- This is a normal consequence of Gauge Invariance indeed

Remark 1

$\mathcal{L}$ is (nearly) invariant under gauge transformation

- $A^\mu \to A^{\mu\prime}(x) = A^\mu(x) - \partial_\mu \theta(x) \qquad \forall \theta(x)$
- $F_{\mu\nu} \to F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = F_{\mu\nu} \underbrace{(-\partial_\mu \partial_\nu \theta + \partial_\nu \partial_\mu \theta)}_{=0}$
- $J^\mu \to J^{\mu\prime} = J^\mu$

Remark 2

$$(5) \Rightarrow \quad \partial_\nu J^\nu = \partial_\nu \partial_\mu F^{\mu\nu} = \Box(\partial_\nu A^\nu) - \partial_\nu \partial^\nu (\partial_\mu A^\mu) = 0$$

has to be if EoM

$\longrightarrow J^\nu$ has to be a CONSERVED CURRENT

conversely if $\partial_\mu J^\mu = 0$, then $\quad A'_\mu J'^\mu - A_\mu J^\mu = \cancel{A_\mu J^\mu} - \partial_\mu(\theta) J^\mu - \cancel{A_\mu J^\mu} = -\partial_\mu(\theta J^\mu) + \theta \underbrace{\cancel{\partial_\mu J^\mu}}_{=0}$

$\to$ Gauge transformation changes $\mathcal{L}$ by a total derivative (wich don't change EoM)

$\to$ If A is a solution, then A' is also a solution

3.2 Coulomb Gauge (transverse spatial propagator and projector)

To see clearer the effects of transverse condition and gauge transformation, let's write

$$A^\mu(k) = \varepsilon^\mu_{(1)} a_{(1)} + \varepsilon^\mu_{(2)} a_2 + k^\mu a_3 + k^\mu_p a_4$$

Using the new basis

where $k^\mu_p = (k^0, -\vec{k})$, parity - flipped of $k^\mu = (k^0, +\vec{k})$

Notice

- $k^\mu + k_P^\mu = (2k^0, \vec{0})$, this decomposition assumes a certain rest frame because the Parity Flipped k_P^μ depend on the Refer. Frame !

- If $k^2=0 \Rightarrow k_P^2 = 0$ also, and $k\cdot k_P = k^{0\,2} + \vec{k}^2 = 2\vec{k}^2 > 0$

- For $i=1,2$: $\underset{\substack{\Updownarrow \\ \varepsilon_\lambda^\mu \quad \lambda=\pm 1}}{\varepsilon_i^\mu k_\mu} = 0 = \varepsilon_{(i)}(k_P)_\mu \Rightarrow \varepsilon_{(i)}^0 = 0$ and $\vec{\varepsilon}_{(i)}\cdot\vec{k} = 0$
Transverse Spatial

Gauge Transformation: $A_\mu \to A_\mu' = A_\mu - \partial_\mu\theta \Leftrightarrow \hat{A}_\mu' = \tilde{A}_\mu + i k_\mu \hat{\theta} \Leftrightarrow \boxed{a_3' = a_3 + i\hat{\theta}}$

we are just touching the THIRD COMPONENT!

Consequence $\Rightarrow$ a_3 is NOT FIXED BY EoM

but maybe we can fix a_3 by imposing the Transverse Condition... (like we did with Proca)

Transverse Condition : $k_\mu \tilde{A}^\mu = 0 = k^2 a_3 + k\cdot k_P a_4$

$\Rightarrow$ Fixes Mainly a_4 (and not a_3, i.e. when $k^2=0$) You DON'T FIX a_3

This means that $\to$ Doesn't Fix $\hat{\theta}$ in general

To have a DEFINITE SOLUTION, use better Gauge Fixing : Colomb Gauge

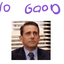
(6) $\boxed{\partial_i A^i(x) = 0} \Leftrightarrow k^i \tilde{A}_i = 0$

WHY IS BETTER?

✲ there is NO $\partial_0 A^0$ term in $\mathcal{L}$ or (5) $\Rightarrow$ A^0 has no Kinematic Energy

Consequence:

(Rembember: would be bad to have it because we will end up with a minus sign in the Kinetic $\to$ NO GOOD)



$\Rightarrow$ A^0 is Fixed locally in time (Fixed time by time)

Equation of motion For A^0: $\Delta A^0 - \partial^0(\partial_i A^i) = J^0$

$\to$ in this gauge $A^0 \equiv 0$ if $J^0 = 0$ OR if $J^0 \neq 0$, it will be Fixed by colomb equation

$\Delta A^0 = J^0$ moment after moment $\left(\begin{matrix} A^0(t\to\infty)\to 0 \\ \text{if } J^0(t\to\infty)\to 0 \end{matrix}\right)$

* Can prove that any solution of (5) can be brought to (6), by a well defined and unique $\theta(x)$

$\Rightarrow$ Obtain well defined propagator and projector

Projector to the TRANSV TO THE SPATIAL VECTOR $\vec{k}$.

$$(P_{TS})^\mu{}_\nu = \sum_{\lambda=1,1} \varepsilon^\mu_{(\lambda)} \varepsilon^{\nu *}_{(\lambda)} = \begin{cases} \delta^{ij} - \dfrac{k^i k^j}{|\vec{k}|^2} & i,j = 1,2,3 \\ 0 & \mu \text{ or } \nu = 0 \end{cases} = -\eta^{\mu\nu} + \frac{k^\mu k_\rho + k^\nu k^\mu_\rho}{k \cdot k_\rho}$$

Transverse Spatial

(For $k^2 = 0$)

$\Rightarrow (P_{TS})_\nu$ IS NOT Lorentz Invariant / Covariant, BUT HAS 2 propagating degrees of Freedom

Can be used to define $\mathcal{M}^{-1}$ and Feynmann propagator

$$(\Delta_F)_{TS} = i\,(G_F)^\mu_\nu(k) = i\,(\mathcal{M}^{-1})^\mu_{TS,\nu} = \frac{i}{k^2 + i\varepsilon}\,(P_{TS})^\mu{}_\nu$$

And the Green Func is the one you get when compute classical response to source $\tilde{J}^\nu(x)$:

$$\tilde{A}^\mu(x) = G_{TS}(x)^\mu{}_\nu\, \tilde{J}^\nu(x) \qquad \text{NOT LORETZ INVARIANT}$$

BUT can show that $\exists$ compensating gauge transformation (For well defined $\tilde{\theta}$)

$$\underbrace{\Lambda^{\mu'}{}_\mu}_{\text{Lorentz Transf}} \hat{A}^\mu(k) = G_{TS}(\Lambda k)^{\mu'}{}_{\nu'}\, \Lambda^{\nu'}{}_\nu\, \tilde{J}^\nu(x) + \underbrace{i\,\Lambda^{\mu'}{}_\mu\, k^\mu\, \hat{\theta}(k)}$$

3.3 Covariant Propagator

To preserve Lorentz Covariance, add a "Gauge Fixing" term:

$$\boxed{\mathcal{L}_{GF} = \frac{1}{2\xi}(\partial_\mu A^\mu)^2}$$

Lorentz Invariant
gauge: NOT INVARIANT
$\partial_\mu A^\mu \to \partial_\mu A'^\mu = \partial_\mu A^\mu - \Box\theta(x)$

$\Rightarrow$ M is NOW invertable:

$$\frac{\delta S}{\delta A_\nu(x)} = 0 \to \Box A^\nu - \partial^\nu(\partial_\mu A^\mu)\left(1 - \frac{1}{\xi}\right) = J^\nu$$

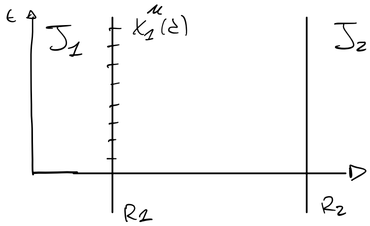
$$\overset{\text{Four Trans}}{\Longleftrightarrow} \underbrace{-k^2\left(\delta^\nu_\mu - \frac{k^\nu k^\mu}{k^2}\left(1 - \frac{1}{\xi}\right)\right)} \tilde{A}^\mu(k) = \tilde{J}^\nu(k)$$

$$M^\nu{}_\mu = -k^2\left(P_T(k) + \frac{1}{\xi}P_L(k)\right)^\nu_\mu$$

$$\Rightarrow (\Delta_F)^\mu_\nu = i\,G^\mu_\nu = i\,(M^{-1})^\mu_\nu = \frac{-i}{k^2 + i\varepsilon}\left(\delta^\mu_\nu - (1-\xi)\frac{k^\mu k_\nu}{k^2}\right) = i\frac{1}{-k^2}(P_T + \xi P_L)^\mu_\nu$$

Response to $\tilde{J}^\nu(k)$ is $\tilde{A}^\mu = G^\mu_\nu \tilde{J}^\nu$ depends on ξ if $k_\nu \tilde{J}^\nu \neq 0$

3.4: Coulomb Potential for static sources



We want to have $\frac{1}{r}$ potential

Looking at the sources:

$$J^\mu(x) = J_1^\mu + J_2^\mu$$

- $J_1^\mu(x) = \int \underset{\text{P.Time}}{d\tau}\; q_1 \frac{dx_1^\mu}{d\tau} \delta^\mu(x - x_1(\tau))$; $x_1^\mu(\tau) = (\tau, \underset{\text{FIXED}}{\vec{R}_1})$

- $J_2^\mu(x) = \int d\tau\; q_2 \frac{dx_2^\mu}{d\tau} \delta^\mu(x - x_2(\tau))$

Fourier Transformation $\rightsquigarrow$ $\tilde{J}_1^\nu(K) = q_1 \int \frac{d^4x}{(2\pi)^4} e^{iK_0x^0 - i\vec{K}\vec{x}} \delta_0^\nu \underbrace{\int d\tau\, \delta(x^0 - \tau)\delta^3(\vec{x} - \vec{R}_1)}_{=1}$

$$= q_1 \delta_0^\nu \delta(K_0) \frac{1}{2\pi} e^{-i\vec{K}\cdot\vec{R}_1}$$

Using the ξ-gauge (covariant)

$$G^\mu{}_\nu(K) = -\frac{1}{K}\left[\delta^\mu_\nu - (1-\xi)\frac{K^\mu K_\nu}{K^2}\right]$$

because of $\delta_0^\nu \Rightarrow K_0 = 0$ so $= 0$ $\Longrightarrow$ ξ independent

$$\Longrightarrow \tilde{A}_1^\mu = G^\mu{}_\nu(K)\tilde{J}_1^\nu(K) = -\frac{1}{-\vec{K}^2}\left[\delta^\mu_\nu - (1-\xi)\frac{K^\mu K_\nu}{K^2}\right]\tilde{J}_1^\nu(K)$$

$$= \frac{q_1}{2\pi \vec{K}^2}\delta_0^\mu \delta(K^0) e^{-i\vec{K}\vec{R}_1}$$ $\Longrightarrow$ Only $\hat{A}_0$ is $\neq 0$

$A_\mu^{\text{Classical}}$

Given the 2 sources, i know the field $\underline{A_\mu^{class}}$, what's the action?

$$\Rightarrow W_{cl}(J) = S(A_\mu^{class}, J_\nu^{given})$$
$$= \int d^4K \left[\frac{1}{2} \tilde{A}_\mu^*(K) \underbrace{M^\mu{}_\nu(K)\tilde{A}(K)}_{\tilde{J}^\mu \text{ by definit}} - \tilde{A}_\mu^*(K)\tilde{J}^\mu(K) \right]$$
$$= -\frac{1}{2}\int d^4K \; \tilde{A}_\mu^*(K)\tilde{J}^\mu = -\frac{1}{2}\int d^4K \; \underline{J_\nu^* G^\nu_\mu} J^\mu \quad = \int dt \; \underbrace{K - V}_{L}$$

But since $\tilde{J} = \tilde{J}_1 + \tilde{J}_2$, we will get

$$\text{"="} \underbrace{-\frac{1}{2} J_1 G J_2 - \frac{1}{2} J_2 G J_1}_{= -J_2 \underline{G J_1}_{\tilde{A}_1}} \underbrace{\underbrace{- \frac{1}{2} J_1 G J_1}_{\text{infinite self energy}} \underbrace{- \frac{1}{2} J_2 G J_2}_{\text{infinite self energy}}}_{\text{Discussed Later}}$$

= How source J_2 feels A_1 created by J_1
(or vice versa)

$$\text{"} -J_2 G J_1 \text{"} = -\int d^4K \; \tilde{J}_2^\mu(K)\tilde{A}_{1\mu}(K)$$

Intermediate
$$= -\underbrace{\int dK^0 \, \delta(K^0)\delta(K^0)}_{=1} \times q_1 q_2 \int \frac{d^3K}{(2\pi)^3} \frac{1}{\vec{K}^2} e^{-i\vec{K}(\vec{R}_2 - \vec{R}_1)}$$
$$= -\frac{1}{2\pi}\int dt \, e^{it(t^0=0)} \cdot \frac{q_1 q_2}{2|\vec{R}_1 - \vec{R}_2|} \qquad \text{Fourier}$$
$$= \int dt \quad V(\vec{R}_1, \vec{R}_2, \cancel{t}) \qquad V(R_1, R_2) = \frac{q_1 q_2}{4\pi |R_2 - R_1|}$$

NOT DEP. on t

$\Rightarrow$ Concrete example of using STRANGE PROPAGATOR and get Coulomb!

IV. QED Lagrangian and Gauge Principle

4.1 QED Lagrangian

Minimal coupling $\Rightarrow$ Starting from Dirac Lagrangian $\mathcal{L} = \bar{\psi}_\beta(x)(i\gamma^\mu)^\beta{}_\alpha \partial_\mu - m)\psi^\alpha(x)$

Replace $\partial_\mu \to D_\mu = (\partial_\mu + ie A_\mu(x))$

In the end you get that the addition of the extra term corresponds to adding a source

$$\mathcal{L}_J = -A_\mu(x) J_D^\mu(x) \qquad \text{with} \qquad J_D^\mu(x) = e\,\bar{\psi}(x)\gamma^\mu\psi(x)$$

↳ conserved Dirac current $\partial_\mu J_D^\mu = 0$

$$\boxed{\Rightarrow \mathcal{L}_{QED} = \bar{\psi}[i\gamma^\mu \partial_\mu + ieA_\mu) - m]\psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \underbrace{- A_\mu J^\mu}_{\text{some other sources}}}$$

Feeling from where this comes from:

4.2 Local Symmetry and Gauge Principle

Dirac $\to$ QED

$$\mathcal{L}_{CKG} = (\partial_\mu \varphi^*)(\partial^\mu \varphi) - m^2 \varphi^* \varphi \qquad \varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$$

$$= \frac{1}{2}(\partial_\mu \varphi_1)^2 + \frac{1}{2}(\partial_\mu \varphi_2)^2 - \frac{m^2}{2}(\varphi_1^2 + \varphi_2^2)$$

Since they have the SAME MASS there is a GLOBAL SYMMETRY $U(1) = SO(2)$

$$S_G: \varphi(x) \to \varphi'(x) = e^{ie\theta}\varphi(x) \qquad \text{and } \theta \text{ DOES NOT DEPEND on } x$$

it's the same for all x

$$\varphi^*(x) \to \varphi^{*\prime}(x) = e^{-ie\theta}\varphi^*(x)$$

$$\varphi^*\varphi \to \varphi^{*\prime}\varphi' = \cancel{e^{+}}\,\cancel{e^{-}}\,\varphi^*\varphi \qquad \text{INVARIANT}$$

Remark: if $\theta \ll 1$: $\varphi' \simeq \varphi + \delta\varphi$, $\begin{cases} \delta\varphi = ie\theta\varphi \\ \delta\varphi^* = -ie\theta\varphi \end{cases}$ it's a small rotation in $\mathbb{C}$

This is equivalent in the real representation of

$$\begin{pmatrix} \delta\varphi_1 \\ \delta\varphi_2 \end{pmatrix} = \begin{pmatrix} -e\theta\varphi_2 \\ +e\theta\varphi_1 \end{pmatrix} = ie\theta \underbrace{\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}}_{T=T^{+}=\tau_2} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

Using this global symmetry WE CAN FIX the phase of φ (since there is this invariance) at one point $x = x_0$

Is it possible to generalize this to a LOCAL symmetry ?

$$\varphi(x) \longrightarrow \varphi'(x) = e^{ie\theta(x)} = \varphi(x)$$

Problem: $\mathcal{L}_{CKG}$ is not invariant under S_L because $|\partial_\mu\varphi|^2$ EXERCISE.

It's not a surprise! $\partial_\mu \varphi_i(x) = \lim_{dx^\mu \to 0} \frac{\varphi_i(x+dx^\mu) - \varphi_i(x)}{dx^\mu}$

$\varphi_i(x+dx) =$ components of φ in the basis at $x+dx$
$\varphi_i(x) =$ φ " " at x

$\rightarrow$ The difference is a difference of 2 vectors in 2 different basis !

DISASTER

$\Longrightarrow$ We NEED to KNOW the change of basis matrix B between the 2:

$$\underbrace{B(x,x+dx)}_{2\times 2 \text{ matrix}} \underbrace{\varphi(x+dx)}_{\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}} = \Big|\varphi(x+dx)\Big|_{\text{in basis at } x+dx} \Big|_{\text{in basis at } x}$$

We can construct the COVARIANT DERIVATIVE :

$$D_\mu \varphi = \lim \frac{B(x,x+dx)\,\varphi(x+dx) - \varphi(x)}{dx^\mu}$$ by def a vector at x $\rightarrow$

S_L : $\xrightarrow{S_L} e^{ie\theta(x)} = D_\mu\varphi(x)$ like φ!

$$\Longrightarrow \mathcal{L}_{CKGL} = |D_\mu\varphi|^2 - m^2|\varphi|^2$$

$\overset{!}{=}$ invariant under S_L $\gg \mathcal{L}_{CKG}$ if $B \equiv I$ $(D_\mu \to \partial_\mu)$

What is the basis? ($\Leftrightarrow$ how is B defined ?)

At each x you need to give youself

$$B(x, x+dx) = 1 + ie\underbrace{A_\mu(x)\,dx^\mu}_{d\theta} + o(dx^2) \qquad \text{(in complex notation)}$$

For any point x, I need $A_\mu(x)$ = connection, connecting base at x and at neighboring point $x+dx$

Connection with General Relativity

Link with $\Gamma^\rho_{\mu\nu}$ from General Relat.: go to matrix notation:

$$B^i{}_j = (I)^i{}_j + ie\,A_\mu\,dx^\mu\,T^i{}_j \quad \overset{\text{equivel}}{\Longleftrightarrow} \quad B^\rho{}_\sigma = \delta^\rho{}_\sigma + \Gamma_\mu{}^\rho{}_\sigma\,dx^\mu$$

So $(\Gamma_\mu)^i{}_j = ie\,A_\mu T^i{}_j = e\,A^i_{\mu j}$ $\quad (\rho,\sigma)$ in tangent space for G.R.

$\rightarrow (i,j)$ in internal space for gauge theories

Little recap...

Matrix of coordinate change

$$B(x,x+dx) = 1 + ieA_\mu(x)dx^\mu + O(dx^2)$$

Introduced the concept of covariant derivative

φ: scalar field
$$D_\mu \varphi = (\partial_\mu + ieA_\mu(x))\varphi(x)$$
$$\xrightarrow{SL} e^{ie\theta(x)} D_\mu \varphi(x)$$

And the field
$$\varphi \xrightarrow{SL} e^{ie\theta(x)}\varphi(x)$$
$$A_\mu \xrightarrow{SL} A'_\mu(x) = A_\mu(x) - \partial_\mu\theta(x)$$
$$B\varphi \xrightarrow{SL} M'\varphi' = e^{ie\theta(x)} B\varphi$$

Explaination Local Transf

$$B(x,x+dx) \xrightarrow{SL} B' = e^{ie\theta(x)} B(x,x+dx)\, e^{-ie\theta(x+dx)} \times \underbrace{e^{ie\theta(x+dx)}\varphi(x+dx)}_{\varphi'}$$

$$B' = e^{ie\theta(x)}(1+ieA_\mu dx^\mu)\, e^{-ie[\theta(x)+\partial_\mu\theta(x)dx^\mu]}$$
$$= (1+ieA_\mu(x)dx^\mu - ie\,\partial_\mu\theta(x)dx^\mu)$$
$$\stackrel{!}{=} (1+ieA'_\mu(x))$$

In the end $A'_\mu = A_\mu - i\partial_\mu\theta$

Consequences?

If $\partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu} = 0$ then

(Not shown $\Rightarrow$) $A_\mu(x) = \partial_\mu\Lambda(x)$ and you can choose a basis in where $\theta(x) = \Lambda(x)$

so that $A'_\mu \equiv 0$ $\Rightarrow$ Trivial Connection $\gg$ $D_\mu\varphi = \partial_\mu\varphi$

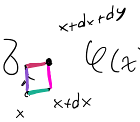
So $F_{\mu\nu} = 0$ is the necessary sufficient condition to find a global basis.

$\rightsquigarrow$ Gauge Invariance

Meaning of this condition?

We we do a closed loop:

Variation of a closed loop



end

coming

$$\delta_{\square}\,\varphi(x) = M(x, x+dx)\, M(x+dx, x+dx+dy) \times$$
$$M(x+dx+dy, x+dy)\, M(x+dy, x)\, \varphi(x) - \varphi(x)$$
$$= ie F_{\mu\nu}(x)\, dx^{\mu} dy^{\nu}\, \varphi(x) + o(dx^3)$$

This $\delta_{\square}$ transforms as the field!

Then

$$\boxed{F'_{\mu\nu} = F_{\mu\nu}}$$

Link with General Relativity

Riemann Tensor $\qquad R^{i}_{\ j\mu\nu} = i\, T^{i}_{\ j}\, F_{\mu\nu}$

Also $\qquad ie F_{\mu\nu}\,\varphi = [D_{\mu}, D_{\nu}]\,\varphi = ie\,(\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu})\,\varphi$

Now going to SCALAR QED

Instead of imposing $F_{\mu\nu} = 0$ as a by hand conditions on the set of A_{μ} fields we take A_{μ} as arbitrary field seriously (physical) $\Rightarrow$ will be determined by kinetical term in a Lagrangian that is gauge invariant (impose S_2)

$$\longrightarrow \quad \mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

in a extremal action:

$$\mathcal{L}_{\substack{QED\\ \text{SCALAR}}} = |D_{\mu}\varphi|^2 - m|\varphi|^2 - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} - J^{\mu} A_{\mu} \qquad \text{(imposing } S_2\text{)}$$

Invariant under

$$\varphi'(x) = e^{i\theta(x)}\varphi(x)$$

$$A'_\mu(x) = A_\mu(x) - \partial_\mu \theta(x)$$

Case of QED

We start from the Dirac Lagrangian

$$\mathcal{L}_{Dirac} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

$$\rightarrow \partial_\mu \psi \rightarrow D_\mu \psi = (\partial_\mu + ieA_\mu(x))\psi(x)$$

Corresponds to $\quad -A_\mu J^\mu_D \quad$ with $\quad J^\mu_D = e\bar{\psi}\gamma^\mu\psi$

($\partial_\mu J^\mu_D = 0$ for solutions of Dirac Equation)

$$\Rightarrow \quad \mathcal{L}_{QED} = \bar{\psi}\left[i\gamma^\mu(\partial_\mu + ieA_\mu) - m\right]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu$$

V. A simple QED (finite) loop: Anomalous Magnetic Moment

a) Introduction

We have to introduce RENORMALIZATION. Why?

Electrostatic self energy of e^-

$$m_{phys} = m_0 + m_{E-S} = m_0 + \frac{4e^2}{5r_e}$$

in $\mathcal{L}$

For $r \to 0$ This term $\to \infty$

For a sphere of radius r_e

(these are the $J_1 G J_1$ and $J_2 G J_2$ terms of lecture 5...)

Non Relativistic Quantum Mechanics

$$H = H_0 + \lambda V$$

$$H^0 |n^{(0)}\rangle = E_n^0 |n^{(0)}\rangle \qquad \text{w/ normalization} \qquad \langle n^{(0)}|m^{(0)}\rangle = \delta_{nm}$$

Perturbation theory

$$|n\rangle = |n^{(0)}\rangle + \lambda |n^{(1)}\rangle + \lambda^2 |n^{(2)}\rangle + \dots$$

Then

$$\langle n|n\rangle = Z_n^{-1} \neq 1 \qquad \text{so it's not normalized!}$$

You have to renormalize them:

$$|n^R\rangle = Z_n^{1/2} |n\rangle \qquad \text{normalization wave functions}$$

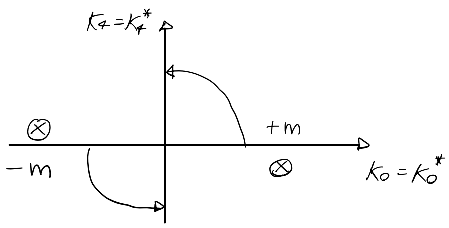
Also we need REGULARIZATION

REGULARIZATION: necessary to shift e.g. m_0 by a (de) finite $-\frac{4}{5}\frac{e^2}{r_e}$ before taking the limit $r_e \to 0$ (where $\infty - \infty$ is ill-defined)

For loops in QFT, e.g. Feynman Propagator $i\Delta_F(\overset{=0}{\widetilde{x-x}}) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon}$

We can:

1) Wick Rotation: defining $k_0 = i k_4$



You can deform the REAL k_0 integral to real $\boxed{k_4 = i k_0}$ without hitting any poles

$$k^2 = k_0^2 - \vec{k}^2 = -k_4^2 - \vec{k}^2 = -k_E^2 = -k_1^2 - k_2^2 - k_3^2 - k_4^2$$

$$k_E = (k_1, k_2, k_3, k_4)$$

then

$$i\Delta_F(0) = \int dk^0 \int \frac{d^3\vec{k}}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} = \int dk_4 \int d^3\vec{k} \, \frac{i \times i}{-k_E^2 - m^2} = \int d^4k_E \, \frac{1}{k_E^2 + m^2} \quad \text{NO POLES}$$

$$= S_3 \int_0^{\cancel{\infty}\,\Lambda} k^3 \frac{1}{k^2 + m^2} \simeq \frac{\Lambda^2}{2} S_3 + O\left(\frac{m^2}{\Lambda^2}\right) \xrightarrow[\Lambda \to \infty]{} \infty$$

(S_{d-1} = surface of $(d-1)$ dimensional sphere, computed by Gaussian integration

$$= \int_{-\infty}^{+\infty} d^4k \, e^{-\frac{1}{2}k^2} = \left(\int_{-\infty}^{+\infty} dk_1 \, e^{-\frac{k_1^2}{2}} \right)^d = \left(\sqrt{2\pi}\right)^d$$

$$= S_{d-1} \int_{-\infty}^{+\infty} dk \, k^{d-1} e^{-\frac{1}{2}k^2}$$

$$\underset{t = \frac{k^2}{2}}{=} \int_0^{\infty} dt \, e^{-t} (2t)^{\frac{d-2}{2}} = S_{d-1} \, 2^{\frac{d}{2} - 1} \Gamma(d/2)$$

Hence

$$S_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

Remember $\quad \Gamma(x) \doteq \int_0^\infty dt\, t^{x-1} e^{-t} = (x-1)\,\Gamma(x-1) = (x-1)! \quad$ For $x \in \mathbb{N}$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

So $\quad S_0 = \frac{2\cancel{\pi^{1/2}}}{\cancel{\sqrt{\pi}}} \quad , \quad S_1 = 2\pi \ \checkmark \quad , \quad S_2 = \frac{2(\pi)^{3/2}}{\frac{1}{2}\Gamma(1/2)} = 4\pi \ \checkmark$

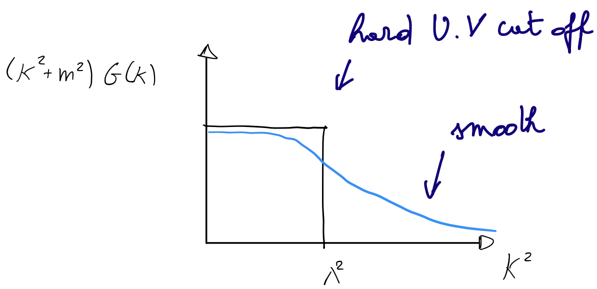
(circle) $\qquad$ (sphere)

2) CHOOSE REGULARIZATION METHOD

2.a) Introduce a ULTRA-VIOLET cut-off Λ in all integrals:

either hard $\quad K_E^2 < \Lambda^2 \quad$ or smooth " " $\quad G(k) = \frac{1}{K^2 + m^2} \longrightarrow$

$$\longrightarrow G(K,\Lambda) = \frac{e^{-(K^2+m^2)/\Lambda^2}}{K^2+m^2} = \int_{1/\Lambda^2}^{\infty} ds\, e^{-s(K^2+m^2)}$$



* OK For 1 loop, complicated for more

* NOT GAUGE INVARIANT

2b) Pauli - Villars

$$\frac{1}{k^2+m^2} \longrightarrow \frac{1}{k^2+m^2} + \sum_i \frac{a_i}{k^2+M_i^2}$$

$$\simeq \frac{1}{k^2+m^2} + O\left(\frac{a_i}{M_i^2}\right) \qquad k^2 \ll M_i^2 \underset{\to\infty}{}$$

$$\simeq \frac{1}{k^2}\Bigg(1+\underbrace{\sum_i \Big(1+\sum a_i}_{=0}\Big)\Bigg) - \frac{1}{k^4}\left(m^2+\sum_i a_i M_i^2\right)$$

$$k^2 \gg M_i^2 \gg m^2$$

3b) Lattice: make S-T discrete

4b) Dimension Regularization

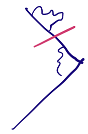
b) Anomalous Magnetic Moment : calculations

(Ref: Peskin - Schröder 6.2 § 6.3)

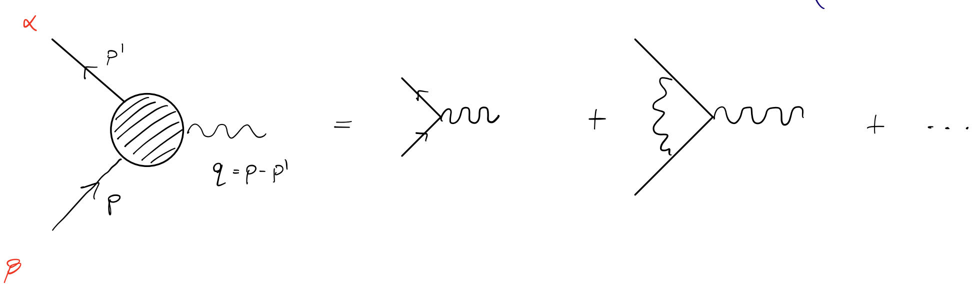
$$\mathcal{L}_{QED} = \bar\psi\left((i\gamma_\mu\partial^\mu + ieA_\mu) - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu(x)\right)^2 - A_\mu(x)J^\mu(x)$$

We can look at loop correction to the $\bar\psi - \psi - A$ $(e - e \cdot \gamma)$ vertex

→ Only 1 Particle Irreducible diagrams

(Not 1PI  cut)

$$-ie\Gamma^\mu(p,p') = -ie\underbrace{\gamma^\mu}_{\Gamma^\mu_{(0)}} - ie\Gamma^\mu_{(1)}(p',p)$$



$\Gamma^\mu(p',p)$ is a 4-vector

If on-shell $p'^2 = p^2 = m^2$, only 1 scalar $p \cdot p' = -\frac{1}{2}p^2 + m^2$

→ Take $q^2 \neq 0$ (<0 for p and p' on-shell)

Also $(\Gamma^\mu)^\alpha{}_\beta$ is a 4x4 matrix in Dirac Space → linear combination of

$$I, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$$

$$1 + 1 + 4 + 4 + 6 = 16$$

Forbidden by Parity

$$\begin{cases} p^\mu \xrightarrow{P} p_P^\mu = (p^0, -\vec{p}) \\ \psi \longrightarrow \gamma^0\psi = \psi_P \end{cases}$$

So in general we can write the general expression of Γ

$$\Gamma^\mu(p',p) = A(q^2)\gamma^\mu + B(q^2)p^\mu + C(q^2)p'^\mu + D(q^2)\sigma^{\mu\nu}p_\nu + E(q^2)\sigma^{\mu\nu}p'_\nu$$

Gauge invariance requires that

$$A_\mu(q) = -iq_\mu \theta(q^2)$$

$$\Rightarrow \underbrace{\bar{u}(p')\Gamma^\mu u(p)}_{J^\mu_{eff}}\, q^\mu = 0$$

$$J^\mu_{eff} \Leftrightarrow \partial_\mu J^\mu_{eff} = 0$$

invariant conservation

$$\Rightarrow A\ \bar{u}'(\cancel{p}-\cancel{p}')u + \bar{u}'u\,(B\,p.(p'-p) + C\,p'.(p'-p)] +$$

$$\bar{u}'\sigma^{\mu\nu}u\,[(p'-p)_\mu(D p_\nu + E p'_\nu)] \equiv 0$$

$$\Rightarrow \cancel{p}u = mu$$

$$\Rightarrow \bar{u}'\cancel{p}' = u'm$$

So in the end

$*\quad A\,\bar{u}u\,(m-m) = 0\qquad \checkmark$ on-shell

$$*\quad \bar{u}'u\,\underbrace{(B-C)}_{B=C}\,\underbrace{(p'p - m^2)}_{\text{Always}} = 0$$

From current conservation

$$*\quad \bar{u}'\sigma^{[\mu\nu]}u\,(p'^\mu p^\nu D - \cancel{p_\mu p_\nu}D + p'_\mu p'_\nu E - \cancel{p_\mu p'_\nu}E - p_\mu p'_\nu E) = 0$$

$$\Rightarrow E = -D$$ from current conservation.

$$J^\mu(q) = J^{\mu*}(-q) \qquad \text{reality condition}$$

$$\Longrightarrow \quad A = A^* \quad , \quad B = B^* \quad , \quad D = -D^*$$

Then, the most general form we would have is then

$$\boxed{\Gamma^\mu(p', p) = A(q^2)\gamma^\mu + B(q^2)(p'^\mu + p^\mu) + D(q^2)\sigma^{\mu\nu}q_\nu} \quad (1)$$

But we can simplify...

Gordon Decomposition

$$F^\mu = \bar{u}'(p')(\underline{\underline{\not{p}\gamma^\mu}} + \underline{\underline{\gamma^\mu \not{p}}})u(p) \qquad (*)$$

$$\overset{!}{=} \bar{u}'(p')(m\gamma^\mu + \gamma^\mu m)u(p) \qquad \text{only true on shell for } u, \bar{u}$$

$$\overset{!}{=} 2m\,\bar{u}'(p')\gamma^\mu u(p)$$

On the other hand we can write

$$\underline{\underline{\gamma^\mu \not{p}}} = \gamma^\mu\gamma^\nu p_\nu = \left(\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu]\right)p_\nu$$

$$\overset{!}{=} (\eta^{\mu\nu} - i\sigma^{\mu\nu})p_\nu$$

By applying this to $=$ $=$ in equation $(*)$

$$\Longrightarrow F^\mu = \bar{u}'(p')\left((p' + p)^\mu + i\sigma^{\mu\nu}(p' - p)_\nu\right)u(p)$$

Then by identifying $F^\mu = F^\mu$ from both equations

$$\boxed{\frac{\cancel{2m}\,\bar{u}'(p')\gamma^\mu u(p)}{\cancel{2m}} = \bar{u}'(p')\left(\frac{p'^\mu + p^\mu}{2m} + i\sigma^{\mu\nu}\frac{(p' - p)_\nu}{2m}\right)u(p)} \qquad (2)$$

Exercise:

Show that the second term on RH side precisely gives the 2° term of the NON RELATIVISTIC Pauli Hamiltonian for a spin 1/2

$$H_B = \frac{e}{2m}\left(\vec{L} + g_s \vec{S}\right)\cdot \vec{B}(\vec{x})$$

acting on N.Rel. 2 compon. $u(x) = \begin{pmatrix} \xi(\vec{r},t) \\ \frac{\vec{p}\cdot\vec{\sigma}}{m}\xi \end{pmatrix}$ spinor ξ

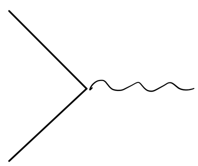
where $\vec{L} = \vec{r}\times\vec{p}$ and $\vec{S} = \frac{\hbar}{2}\sigma$

Hint: Take the square of Dirac equation with external $A_\mu(x)$ leading to static $\vec{B}(\vec{x})$ in the N.R. limit.

$$(1) \xrightarrow{(2)} \Gamma^\mu(p',p) = F_1(q^2)\gamma^\mu + F_2(q^2)\frac{i\sigma^{\mu\nu}}{2m}q_\nu$$

$$= \gamma^\mu \underbrace{(A + 2mB)}_{F_1} + \underbrace{(-2miD - B)}_{F_2}\, i\frac{\sigma_{\mu\nu}}{2m}q_\nu$$

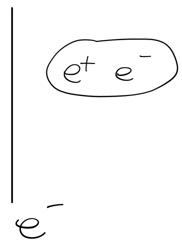
Form factor $F_1(q^2)$ describes the electric charge of e^-, as probed by a virtual "γ" of momentum q^μ



At large distances $(q^\mu \to 0)$, $F_1(0) = 1$ $\to$ Charge $-e$

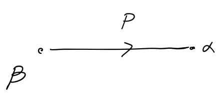
At small distances

$F_2(|\vec{p}|^2 \gg m^2) \nearrow$, the screening by (e^+e^-) in vacuum polarizations go away

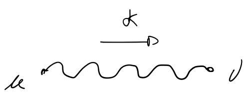


$F_2(0)$ = "magnetic Form Factor" in what we want to compute

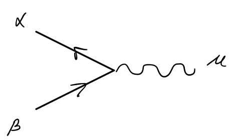
c) QED Rules



$$\left(\frac{i}{\not{p} - m + i\varepsilon}\right)_{\alpha\beta} = \left[\frac{\delta^2(iS)}{\delta\bar{\psi}^\alpha\,\delta\psi_\beta}\right]^{-1}$$



$$\frac{-i\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)}{k^2 + i\varepsilon}$$



$$-ie\,(\gamma^\mu)_{\alpha\beta} = \frac{\delta^3(iS)}{\delta\bar{\psi}^\alpha\,\delta\psi^\beta\,\delta A_\mu}$$

$$(-i)^F \int \frac{d^4k}{(2\pi)^4} \qquad F = 1 \quad (\text{Fermion loop})$$

Exercise

Show that the second term on RH side precisely gives the 2° term of the NON RELATIVISTIC Pauli Hamiltonian for a spin 1/2

$$H_B = \frac{e}{2m}\left(\vec{L} + g_s \vec{S}\right)\cdot \vec{B}(\vec{x})$$

acting on N.Rel. 2 compon. $u(x) = \begin{pmatrix} \xi(\vec{r},t) \\ \frac{\vec{p}\cdot\vec{\sigma}}{m}\xi \end{pmatrix}$ spinor ξ

where $\vec{L} = \vec{r}\times\vec{p}$ and $\vec{S} = \frac{\hbar}{2}\sigma$

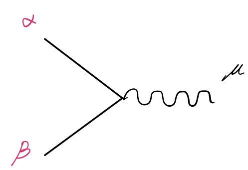
Hint: Take the square of Dirac equation with external $A_\mu(x)$ leading to static $\vec{B}(x)$ in the N.R. limit.

Solution

Let's start from the Lagrangian

$$i\mathcal{L} = \bar{\Psi}(p')\left(-ie\Gamma^\mu(p',p)A_\mu(q)\right)\tilde{\Psi}(p) + \dots$$

For now we consider order 0, and so $\Gamma^\mu_{(0)} = (\gamma^\mu)^\alpha_\beta$, this corresponds to 1° order Feynmann Diagram:



So we will have

$$\frac{\partial\mathcal{L}}{\partial\bar{\Psi}} \longrightarrow i\gamma^\mu\underbrace{\left[\partial_\mu + ieA_\mu(x)\right]}_{\cancel{D}}\Psi(x) - m\Psi(x) = 0 \iff i\cancel{D}\Psi = m\Psi$$

and after a Fourier Transf on $\mathcal{L}$ we get

$$\left[i\cancel{\partial} - \left(F_1(0)\gamma^\mu + F_2(0)\sigma^{\mu\nu}q_\nu\right)A_\mu(x) - m\right]\tilde{\Psi}(p)$$

We are considering the 3-level process, and so

$$\begin{cases} F_1(0) = 1 \\ F_2(0) = 0 \end{cases}$$

Using these 2 $F_{1,2}$ we will have (starting from $i\not{D}\psi = m\psi$)

$$\Rightarrow 0 = (i\not{D}+m)(i\not{D}-m)\psi(x)$$
$$= (-\gamma^\mu\gamma^\nu D_\mu D_\nu - m^2)\psi$$

but

$$\{\gamma^\mu,\gamma^\nu\} = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\} + \frac{1}{2}[\gamma^\mu,\gamma^\nu]$$
$$= \eta^{\mu\nu} - i\sigma^{\mu\nu}$$

Then

$$0 = \left(-\eta^{\mu\nu}D_\mu D_\nu + i\sigma^{\mu\nu} D_\mu D_\nu - m^2\right)\psi$$
$$= \left(-\eta^{\mu\nu}D_\mu D_\nu + \frac{i}{2}\sigma^{\mu\nu}\underbrace{\overset{\text{Antisymmetric}}{[D_\mu,D_\nu]}}_{ieF_{\mu\nu}} - m^2\right)\psi$$
$$= \left(-D^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}(x) - m^2\right)\psi$$

If F_1, F_2 not specified we would have $(F_1(0)+F_2(0)) + \underbrace{\mathcal{O}(F_{\mu\nu})^2}_{\text{we consider lower order}}$

>> But we choose F_1, F_2. so let's consider just the black

For a constant magnetic field applied $\vec{B} = B\vec{e}_z = \vec{\nabla}\times\vec{A}$, then

$$\begin{cases} \vec{A}(x) = \frac{1}{2}(\vec{B}\times\vec{r}) = \left(-\frac{1}{2}By, +\frac{1}{2}Bx, 0\right) \\ A_0 \equiv 0 \end{cases}$$

$$\rightsquigarrow \Big(\overbrace{-\partial_0^2 + (\vec{\partial} - ie\vec{A})^2}^{D^2} - e\sigma^{12}\overset{F^{\mu\nu} \text{ but } \mu=1,\ \nu=2}{F_{12}} - m^2\Big)\psi = 0$$

A solution is $\psi(t,\vec{x}) = e^{-iEt}\varphi(\vec{x})$ with $E = m+\varepsilon$; $\varepsilon \ll m$ $\rightarrow E^2 = m^2 + 2m\varepsilon + \mathcal{O}(\varepsilon^2)$

$\hookrightarrow$ N.R. kinetic energy

Then, since $H\varphi(x) = \mathcal{E}\,\varphi(x)$ with $\mathcal{E}$ eigenvalue (energy of the state)

$$\mathcal{E}\,\varphi(\vec{x}) = -\frac{1}{2m}\left((\vec{\partial} - ie\vec{A})^2 - e\sigma^{12}F_{12}\right)\varphi = H\varphi$$

→ H is the Pauli hamiltonian. Usually H is acting on 2 spinors.

$$H\varphi = -\frac{1}{2m}\left(\vec{\partial}^2 - ie\underbrace{\vec{\partial}\vec{A}}_{=0} - 2ie\vec{A}\cdot\vec{\partial} - \underbrace{e^2\vec{A}^2(x)} - e\sigma^{12}F_{12}\right)\psi$$

Negligeable for small $\vec{A}$

Let's focus on $-2ie\vec{A}\cdot\vec{\partial}$

$$-2ie\vec{A}\cdot\vec{\partial} = -2ie\frac{B}{2}\left(-y\,\partial_x + x\,\partial_y\right) = eBL_z$$

Reminder
$L_z = (p\times r)_z = (\vec{r}\times(-i\hbar\vec{\partial}))_z$
$= (x\hbar(-i\partial_y) - y\hbar(-i\partial_x))$

Let's focus on F^{12}

$$F^{12} = -B_z = -F^{21} = F_{12}$$

(ACTUNG! lowering indices)

$F^{ji} = -\varepsilon^{jik}\underline{B_k}$

3D and so $B_k = B^k$

Then σ^{12}

$$\sigma^{12} = \frac{i}{2}[\gamma^1,\gamma^2]$$
$$= \frac{i}{2}\left[\begin{pmatrix}0 & \sigma_1\\ -\sigma_1 & 0\end{pmatrix},\begin{pmatrix}0 & \sigma_2\\ -\sigma_2 & 0\end{pmatrix}\right]$$
$$= \frac{i}{2}\begin{pmatrix}[\sigma_1,\sigma_2] & 0\\ 0 & [\sigma_1,\sigma_2]\end{pmatrix} = \frac{i}{2}\begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix}$$

IT'S DIAGONAL!

Since it's diagonal we can decompose φ in 2 parts and we will keep the one that will be multiplied by σ:

NOT SURE

$\varphi = \begin{pmatrix}\varphi_+\\ \varphi_-\end{pmatrix}$ φ_+ Big, φ_- small $\sim \frac{\vec{p}\cdot\vec{\sigma}}{m}\varphi_+$ So Just keep the interesting φ_+

?

Hence

$$\mathcal{E}\varphi_+ = \left(-\frac{\vec{\partial}^2}{2m} + \frac{(eB)^2}{8m}(x^2+y^2) + \underbrace{\frac{-e}{2m}BL_z - \frac{eB}{2m}\sigma_3}_{-\mu B} \right)\varphi_+$$

2 component Pauli Spinor $\rightarrow \begin{pmatrix} \sigma_3 = 1 \\ \sigma_z = -1 \end{pmatrix}$

with

$$\mu = \frac{e}{2m}L_z + \frac{e}{2m}\sigma_3 \quad \text{(Magnetic Moment)}$$

$$= \frac{e}{2m}(L_z + g_s S_z) \longrightarrow S_z = \frac{\sigma_3}{2} \quad \text{and} \quad g_s = 2$$

Explaination

So if I prepare my system with an angular momentum $\vec{L}$ NOT in the z direction, there will be a PRECESSION at LARMOR Frequency

$$\omega_L = \frac{eB}{2m} \quad (g_L = 1)$$

Larmor Frequency

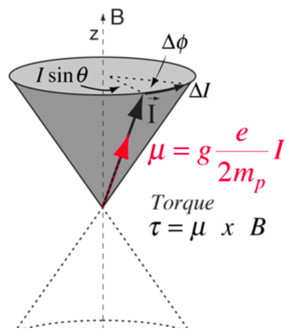
When a magnetic moment is placed in a magnetic field it will tend to align with the field. Classically, a magnetic moment can be visualized as a current loop and the influence toward alignment can be described as the torque on the current loop exerted by the magnetic field. The idea of the magnetic moment as a current loop can be extended to describe the magnetic moments of orbital electrons, electron spins and nuclear spins. In each case the magnetic moment is associated with the angular momentum, and a torque can be identified which tends to align the magnetic moment with the magnetic field. In the nuclear case, the angular momentum involved is the intrinsic angular momentum **I** associated with the nuclear spin.

When you have a magnetic moment directed at some finite angle with respect to the magnetic field direction, the field will exert a torque on the magnetic moment. This causes it to process about the magnetic field direction. This is analogous to the precession of a spinning top around the gravity field. The torque can be expressed as the rate of change of the nuclear spin angular momentum **I** and equated to the expression for the magnetic torque on the magnetic moment

$$\tau = \frac{\Delta I}{\Delta t} = \frac{I\sin\theta\Delta\phi}{\Delta t} = |\mu B\sin\theta| = \frac{ge}{2m_p}IB\sin\theta$$

which when put in derivative form gives a precession angular velocity

$$\omega_{Larmor} = \frac{d\phi}{dt} = \frac{ge}{2m_p}B$$



It can also be visualized quantum mechanically in terms of the quantum energy of transition between the two possible spin states for spin 1/2. This can be expressed as a photon energy according to the Planck relationship. The magnetic potential energy difference is hυ = 2μB. The angular frequency associated with a "spin flip", a resonant absorption or emission involving the spin quantum states is often written in the general form

$$\omega = gB$$

where g is called the gyromagetic ratio (sometimes the magnetogyric ratio). Note that this frequency is a factor of two higher than the one above because of the spin flip with energy change ΔE = 2μB.

So if you place an e^- in a cyclotrone whith a frequency $\omega_{cy} = \frac{eB}{m}$ you will get

$$\omega_s = \frac{g_s eB}{2m} = \frac{eB}{m} = \omega_{cyclotron}$$

$\Rightarrow$ After a Loop in a cyclotrone the spin will be exact in the state . . .

UNLESS g_s is not equal to 2 or we have non-zero ANOMALOUS MAGNETIC MOMENT

$$\mu_{AMM} = \frac{e}{2m}(g_s - 2)S_z \rightsquigarrow a_s = \frac{g_s - 2}{2} \leftarrow F(0) \neq 0$$

So now we have a motivation to redo all the calculation but now we add informations:

1 Loop Corrections

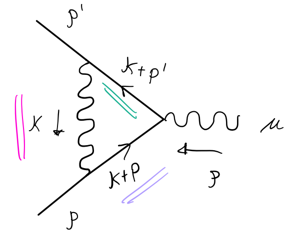
We know the width

$$(-ie)\,\Gamma^\mu = \Gamma^\mu_{(0)} + \Gamma^\mu_{(1)}$$

We factorize this here..

and we used $\Gamma^\mu_{(0)} = (\gamma^\mu)^\alpha{}_\beta$ and so $F_{1(0)} = 1$ independent of p

Now let's see the new $\Gamma^\mu_{(1)}(p', p)$



Using Feynmann Rules, in $\xi = 1$ gauge

here no $-ie$ cause we factorize it before

$$\Gamma^\mu_{(1)}(p', p) = \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2 + i\varepsilon} (-ie\gamma^\nu) \frac{1}{\not{k} + \not{p}' - m + i\varepsilon} (\ )\gamma^\mu \frac{1}{\not{k} + \not{p} - m + i\varepsilon} (-ie\gamma_\nu)$$

$$= -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{N^\mu(k)}{(k^2 + i\varepsilon)((p'+k)^2 - m^2 + i\varepsilon)((p+k)^2) - m^2 + i\varepsilon)}$$

with $N^\mu = \gamma^\nu(\not{p}' + \not{k} + m)\,\gamma^\mu(\not{p} + \not{k} + m)\,\gamma_\nu$

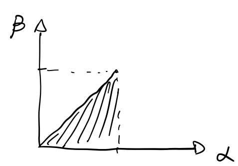
For the Denominator we should use the trick

$$\frac{1}{xyz} = 2 \cdot \int_0^1 d\alpha \int_0^\alpha \frac{1}{D(\alpha, \beta)}\, d\beta$$

why 2?

↓

You can integrate over the full square and remove the 2



Then

$$D(\alpha,\beta) = \left[z + \alpha(x-z) + \beta(y-z)\right]^3 \quad \text{and for us} \quad z = (k^2 + i\varepsilon)$$

Hence

$$D = \left\{k^2 + i\varepsilon + \alpha\left[(p'+k)^2 - m^2 - k^2\right] + \beta\left[(p+k)^2 - m^2 - k^2\right]\right\}^3$$

For large k_E we can revaluate our integral

$$\Gamma^\mu_{(1)} \sim \int^\Lambda dk_E \, k_E^2 \, \frac{k_E^2}{k_E^6} \quad \sim \quad \ln(\Lambda) \to \infty \text{ for } \Lambda \to \infty \qquad \text{DIVERGENT}$$

Interesting! This divergence does not depend on p.

Let's take a closer look at this divergence here:

We can split the VERTEX CORRECTION in an on-shell and off-shell part

$$\Gamma^\mu_{(1)}(p,p') = \Gamma^\mu_{(1)}(p,p) + \left[\Gamma^\mu_{(1)}(p,p') - \Gamma^\mu_{(1)}(p,p)\right] \equiv \underline{\Gamma^\mu_{(1)}(p,p)} + \underline{\Gamma^\mu_{(1)\,off}(p,p')}$$

$$\Longrightarrow \quad \Gamma^\mu_{(1)\,off} = \Gamma^\mu_{(1)}(p',p) - \Gamma^\mu_{(1)}(p,p)$$

Let's now rewrite the Fermion propagator to where the divergence hits:

$$\frac{1}{\not p' + \not k - m} = \frac{1}{\not p + \not k - m + (\not p' - \not p)} = \frac{1}{\not p + \not k - m} - \underbrace{\frac{1}{\not p + \not k - m}}_{\sim k \to \infty \ \frac{1}{k_E^2}} (\not p' - \not p) \underbrace{\frac{1}{\not p + \not k - m}}_{\text{extra power of } \frac{1}{k}} + \cdots$$

Trik used here: $\frac{1}{A+B} = \frac{1}{B} - \frac{1}{B} A \frac{1}{B} + \cdots$

So the first term of this expansion leads to the logarithmic divergence of the loop integral for large k.

IN CONTRAST the reminder of the expansion that vanishes for $p' - p = q \to 0$ contains additional powers of $1/k$ and thus is convergent

Hence the divergence is contained only in the on-shell part of the vertex correction, while the function

$$\Gamma^{\mu}_{(1)\,off}(p,p') = \Gamma^{\mu}_{(1)}(p,p') - \Gamma^{\mu}_{(1)}(p,p) \quad \text{is} \quad \text{FINITE}.$$

MOREOVER, since

$$\Gamma^{\mu}(p,p') = F_{1(1)}(q^2)\gamma^{\mu} + F_{2(1)}(q^2)\underbrace{\frac{i\sigma^{\mu\nu}q_\nu}{2m}}_{\to 0 \text{ for } q_\nu \to 0}$$

we can say that the divergence is confined to $F_{1(1)}(0)$, while $F_{2(1)}(0) = 0$.

→ The divergence is only connected to a quantity already present in the classical Lagrangian, THE ELECTRIC CHARGE.

So let's RENORMALIZE e.

RENORMALIZATION

Let's call $e_{Bare} \equiv e_B$ the coupling constant in $\mathcal{L}_B$

Then

$$e_B\Gamma^{\mu} = e_B(\Gamma^{\mu}_{(0)} + \Gamma^{\mu}_{(1)})$$
$$\overset{!}{=} e_B(\gamma^{\mu}\underbrace{(1 + F_{1(1)})}_{F_1} + F_{2(1)}\sigma^{\mu\nu} + \dots)$$

So we can make it FINITE by choosing
$$e_B = eZ_e = e(1 + F_{1(1)} + \mathcal{O}(e^4))^{-1}$$
$$\overset{!}{\simeq} e(1 - \underbrace{F_{1(1)}}_{} + \mathcal{O}(e^4))$$

↳ ∞ counter term to add to the $\mathcal{L}$

and so we reorganize perturbation theory in $(e)^n$ INSTEAD of $(e_B)^n$.

$$\Rightarrow e_B F_1 \doteq e = \sqrt{4\pi\alpha} \simeq \sqrt{\frac{4\pi}{137}} \quad \text{physical}$$

Let's now do the calculation for the vertex function.
In order to avoid calculations, we restrict ourselves therefore to the part contributing to the magnetic form factor $F_2(0)$.

Because of

$$\Gamma^\mu(p,p') = [F_1(q^2)+F_2(q^2)]\gamma^\mu - F_2(q^2)\frac{(p'+p)^\mu}{2m}$$

we can simplify the calculation of $N^\mu(k)$ by simply NOT CONSIDER terms proportional to γ^μ which do not contribute to the magnetic moment.

Moreover we can consider the limit that the electrons are on shell and so the momentum transfer to the photon $q_\mu \to 0$

* On shell condition is $p^2 = p'^2 = m^2$

$$D = \{ k^2 + i\varepsilon + \alpha(\cancel{k'^2} + \cancel{k^2} + 2k\cdot p' - \cancel{m^2} - \cancel{k^2}) + \beta(\cancel{k^2} + \cancel{k^2} + 2kp - \cancel{m^2} - \cancel{k^2})$$

$$= \{k^2 + 2\alpha\, k\cdot p' + 2\beta\, k\cdot p\}^3$$

Now $\ell^\mu = k^\mu + \alpha p'^\mu + \beta p^\mu$ and so

$$D = \{\ell^2 - (\alpha p' + \beta p)^2\}^3 = \{\ell^2 - (\alpha^2 m^2 + \beta^2 m^2 + 2\alpha\beta\, p'\cdot p)\}^3$$

Since $q \to 0$, $q^2 = 2m^2 - 2p'\cdot p \to 0$ we can replace $p'\cdot p \to m^2$ and so

$$D = \{\ell^2 - (\alpha+\beta)^2 m^2\}^3$$

Now we evaluate the numerator N^μ by changing integration variable: $k = \ell - (\alpha p' + \beta p)$ to ℓ
So

$$N^\mu(\ell) = \gamma^\nu(\not{a}' + \not{\ell} + m)\gamma^\mu(\not{a} + \not{\ell} + m)\gamma^\nu$$

with

$\not{a}' = (1-\alpha)\not{p}' - \beta\not{p}$ $\qquad$ $\not{a} = (1-\beta)\not{p} - \alpha\not{p}'$

Now we want to expand in powers of m

$$N^\mu = -2m^2\gamma^\mu +$$

why? $\Rightarrow$ $\gamma^\nu\gamma^\mu\gamma_\nu = \underbrace{\{\gamma^\nu,\gamma^\mu\}}\gamma_\nu - \gamma^\mu\gamma^\nu\gamma_\nu$

$2\eta^{\mu\nu}\gamma_\nu - \gamma^\mu \frac{1}{2}\underbrace{\{\gamma^\nu,\gamma^\alpha\}\eta_{\nu\alpha}}_{4}$

So it's $\propto \gamma^\mu$ and so cannot contribute to $F_2(0)$

$$+ \; m(\gamma^\nu\{\not{\ell},\gamma^\mu\}\gamma_\nu)$$

It's zero because integration over a symmetric integration

$\Rightarrow \int d^4\ell\, \ell^\mu = 0$

$$+ \; \boxed{\gamma^\nu(\not{Q}'\gamma^\mu + \gamma^\mu\not{Q})\gamma_\nu)} \quad ① \quad + \quad \boxed{\underbrace{m^0}_{=1}(\gamma^\nu\not{\ell}\gamma^\mu\not{\ell}\gamma_\nu)} \quad ②$$

+ (something $\propto \ell$ that $\to 0$ by symmetric integration

$$+ \; \boxed{\gamma^\nu\not{Q}'\gamma^\mu\not{Q}\gamma_\nu} \quad ③$$

if integrate symmetrically on α,β

So we can rewrite ① $\to$ $4m(Q'_\mu + Q^\mu) = 4m\,(1-(\alpha+\beta)(p'^\mu+p^\mu))$

② $\to$ $-4\,\ell^\mu\not{\ell} + 2\ell^2\gamma^\mu \propto \gamma^\mu$ $\to$ Don't care again

③ $\to$ (using $\not{p}' = m$ and $\not{p} = m$ on shell)

$$= 4m(p'^\mu + p^\mu) + \gamma^\mu(4m^2 - 2p^2)$$

Hence

$$N^{\mu} = 2m\left[(1-\alpha-\beta)(\alpha+\beta)(p'^{\mu}+p^{\mu})\right] + O(\gamma^{\mu}) + O(\ell^{m})$$

and so

$$\Pi^{\mu}_{2(1)}(q^2=0) = -2ie^2 \int d\alpha \int d\beta \underbrace{\int \frac{d^4\ell}{(2\pi)^4} \frac{N^{m}}{(\ell^2-(\alpha+\beta)^2 m^2)^3}}_{N^{\mu} I\left(\frac{d}{2},3\right)}$$

We know

$$I(2,3) = \frac{-i}{32\pi^2} \frac{1}{(\alpha+\beta)^2 m^2}$$

Then the result

$$I\left(\frac{d}{2},3\right) = -\frac{e^2}{8\pi^2}\frac{1}{2m}(p'^{\mu}+p^{\mu})$$

$$= -F_{2(0)}(0)\,\frac{p'^{\mu}+p^{\mu}}{2m}$$

$$\Rightarrow F_2(0) = \frac{\alpha}{2\pi} = a = \frac{g_s-2}{2} = 0.001164$$

So we discovered that $g_s \neq 2$ and it's

$$\boxed{g_s = 2\left(F_2(0)+F_2(0)\right) = 2\left(1+\frac{\alpha}{2\pi}+O(e^4)\right)}$$