Homework 1

I.1 Fermion mass in the SM

a) In the SM formulation - a Lagrangian formulation - all the terms must be Lorentz invariant. The quantity $\bar{\Psi}_L \cdot \Psi_L$ is not. (also for R)

Under a Lorentz transformation Λ the LH doublet Ψ_L transform as

$$\Psi_L(x) \to \Psi_L'(x') = S(\Lambda)\, \Psi_L(\Lambda^{-1}x) \qquad \text{w/ } S(\Lambda) \rightsquigarrow \text{Spinor representation of Lorentz. Trans.}$$

For $\bar{\Psi}_L$ we will have

$$\bar{\Psi}_L(x) \to \bar{\Psi}_L(x') = \bar{\Psi}_L(\Lambda^{-1}x)\, S^{-1}(\Lambda)$$

Now the dot product

$$\bar{\Psi}_L \cdot \Psi_L \to \bar{\Psi}_L' \cdot \Psi_L' = \bar{\Psi}_L(\Lambda^{-1}x)\, S^{-1}(\Lambda) S(\Lambda)\, \Psi_L(\Lambda^{-1}x')$$

and since $\Lambda^{-1}\Lambda$ is not necessarily equal to $\mathbb{1}$, this is not invariant.

b) Be a Majorana field would be enough. In that case $f_L = f_L^c$ or $f_R = f_R^c$

I.2 Conservation of Lepton number

Conservation of $L_\alpha \to$ invariance SM under $U(1)$ transformation

a) The transformation is a $U(1)$ transformation that can be parametrized w/ α

$$\chi \to \chi' = e^{i\alpha}\chi \qquad \bar{\chi} \to \bar{\chi}' = e^{-i\alpha}\chi$$

b) I'll denote the conserved current as J^ρ.

Definition:
$$J^\rho = \mathcal{L}\,\delta x^\rho + \frac{\partial \mathcal{L}}{\partial(\partial_\rho \varphi)}\,\delta\varphi$$

The transformation doesn't act on the coordinates, so $\delta x^\rho = 0$

Let's compute $\delta \varphi(x^\rho)$ for both the fields

$$\delta \varphi(x) = \varphi'(x) - \varphi(x) = (e^{i\alpha} - 1)\,\varphi(x) \simeq i\alpha\,\varphi(x)$$

$$\delta \bar{\varphi}(x) \simeq -i\alpha\,\bar{\varphi}(x)$$

also L and R fields in both cases $(\varphi, \bar{\varphi})$ will be the same $\delta\varphi$.

Then

These came from the derivative of the $\mathcal{L}$ w.r.t $\partial_\rho \varphi$ (with the correct φ)

$$J^\rho_\alpha = i\,\bar{\nu}_{\ell L}\gamma^\mu \delta^\rho_\mu\, i\alpha\, \nu_{\ell L} + i\,\bar{\ell}_L \gamma^\mu \delta^\rho_\mu\, i\alpha\, \ell_L + i\,\bar{\ell}_R \gamma^\mu \delta^\rho_\mu\, i\alpha\, \bar{\ell}_R$$

$$= \alpha\,\underbrace{\left[-\bar{\nu}_{\ell L}\gamma^\rho \nu_{\ell L} - \bar{\ell}_L \gamma^\rho \ell_L - \bar{\ell}_R \gamma^\rho \ell_R\right]}_{j^\rho_\alpha}$$

c) The conserved charge will be

$$Q = \int d^3\vec{x}\, j^0 = -\int d\vec{x}\,\left[\bar{\nu}_{\alpha L}\gamma^0 \nu_{\alpha L} + \bar{\ell}_\alpha \gamma^0 \ell_\alpha\right]$$

I.3 Fermion Mixing in the SM : q v.s ℓ

a) From Flavour physics, we can write the quark currents

Charged current : $j_q^{cc} = W_\mu^+ \bar{q}_u \gamma^\mu (1-\gamma^5) q_d$ w/ q_u, q_d up / down quark.

Neutral current : $j_q^{NC} = -Z_\mu \bar{q} \gamma^\mu (1-\gamma^5) q$ w/ q any quark type

b) From the notes

$$j_\ell^{cc} = W_\mu^+ \bar{\nu}_\ell \gamma^\mu (g_L - g_R \gamma^5) \ell$$

$$j_\ell^{NC} = g_e Z_\mu \bar{\ell} \gamma^\mu (g_V - g_A \gamma^5) \ell + g_\nu Z_\mu \bar{\nu}_\ell \gamma^\mu (g_V - g_A \gamma^5) \nu_\ell$$

the structure of the currents are similar for quarks and leptons. In analogy with cc connecting up with Down quarks, cc in leptons connect the charged leptons with their corresponding neutrinos.

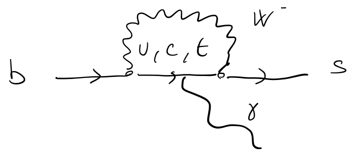
The V-A structure is also similar ⟹ BUT DIFFERENT constants >> Not 100% violation.

c) The lepton sector equivalent of the CKM matrix is the PMNS matrix. It describe the mixing between neutrino Flavors

But since we are talking about the SM in which the ν_s are massless, there is no equivalent for the CKM.

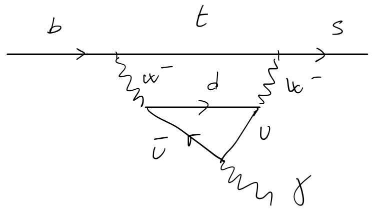
d) $b \to s\gamma$

Assuming that the "spectator quarks" are not represented, we can have



Then it's possible to show that the diagram with the top quark contribute the most, because it's the heavier.

Also this one is possible



e) In the SM the Lepton number is conserved, such diagram breaks the Lepton conservation so it's not possible.

HOMEWORK II

II.1 Neutrino detection via CC, NC, ES.

a) Why NC detection is sensitive to all flavours, while cc only measure $\Phi(\nu_e)$.

NC detection: is based on the fact that Z can decay in all the types of neutrinos and this doesn't depend on the enviroment in which the detector is.

cc detection: For the current detection the answer is that IT DEPENDS.

For detection based on the production of ν's with reactors, since the neutrinos produced are not energetic enough to create leptons heavier than the electron $\Rightarrow$ JUST ν_e are measurable.

If we measure neutrinos coming from space, COULD HAPPEND that high energy muonic, tauonic neutrino are produced and then also $\Phi(\nu_\mu)$, $\Phi(\nu_\tau)$ is measurable.

b) If we consider also charged currents, since the couplings of the currents are $\frac{g_z}{2}$ and $\frac{g_w}{\sqrt{2}}$

$$\frac{g_z^2}{4} \cdot \frac{g_w^{-2}}{2^{-1}} = \frac{g_w^2}{16\cos^2\theta_w} \cdot \frac{2}{g_w} = \frac{1}{8\cos^2(\theta_w)} = 0.17 \qquad (\theta_w \simeq 30°)$$

$$\Rightarrow \quad \sigma^{NC}(\nu_e) = 0.17\,\sigma^{cc}(\nu_e) \quad \rightarrow \quad \sigma^{cc}(\nu_e) = 6\,\sigma^{NC}(\nu_e)$$

$$\sigma(\nu_e) \simeq 6 \times (\sigma(\nu_\mu) + \sigma(\nu_\tau))$$

II. 2. Invariant Formulation d oscillation

Consider the quadratic product $U_4 = U_{\alpha K}^{*} U_{\beta K} U_{\alpha j} U_{\beta j}^{*}$

a) Verify that U_4 is invariant under $\ell_\alpha \to e^{i\varphi_\alpha}\ell_\alpha$

We know $V_L^{\nu\dagger} V^{\ell} = U_{PMNS}$

And also
$$\left.\begin{aligned} \ell_L &= V_L^{\ell\dagger}\,\ell'_L \\ \ell_R &= V_L^{\ell\dagger}\,\ell'_R \end{aligned}\right\} \qquad \overset{\text{MASS}}{\bar{\nu}} = \underbrace{V^{\nu\dagger}}_{U_{PMNS} \text{ if } V_L^{\ell} = \mathbb{1}}\ \overset{\text{Weak}}{\nu}$$

Then element by element $|\nu_\alpha\rangle = \sum_K U_{\alpha K}^{*} |\nu_K\rangle$

Hence
$$U_{\alpha K}^{*} = \langle \nu_\alpha | \nu_K \rangle = e^{-i\varphi_\alpha}\langle \nu_\alpha | \nu_K\rangle e^{i\varphi_K} = e^{-i\varphi_\alpha} U_{\alpha K}^{*} e^{i\varphi_K}$$

Then
$$U_{\alpha K} \to e^{i\varphi_\alpha} U_{\alpha K} e^{-i\varphi_K}$$
$$U_{\alpha K}^{*} \to \left(e^{i\varphi_\alpha} U_{\alpha K} e^{-i\varphi_K}\right)^{*}$$

$$\Rightarrow U_4 = \cancel{e^{-i\varphi_\alpha}} U_{\alpha K}^{*} \cancel{e^{i\varphi_K}}\, \cancel{e^{i\varphi_\beta}} U_{\beta K} \cancel{e^{-i\varphi_K}}\, \cancel{e^{i\varphi_\alpha}} U_{\alpha j} \cancel{e^{-i\varphi_j}}\, \cancel{e^{-i\varphi_\beta}} U_{\beta j}^{*} \cancel{e^{i\varphi_j}}$$
$$\overset{!}{=} U_{\alpha K}^{*} U_{\beta K} U_{\alpha j} U_{\beta j}^{*}$$

b) Show that possibly present Majorana phases do not enter the oscillation probability

$$U \to U \times \mathrm{diag}\left(e^{i\frac{\lambda_\alpha}{2}}\right)$$

The phases cancel out because the U_4 is composed of 2 "U" and 2 "U^{*}", so globally the phase cancel out

$\Rightarrow$ This means that neutrino oscillations processes are blind to the nature of neutrinos.

II.3 Antineutrino Oscillations

Consider the decomposition $|\nu_\alpha\rangle = \sum_k U^*_{\alpha k} |\nu_k\rangle$ $\alpha = e,\mu,\tau$, $\beta = 1,\dots,3$

a) Show antineutrinos can be decomposed as $|\bar\nu_\alpha\rangle = \sum_k U_{\alpha k} |\nu_k\rangle$

We know that $\bar\nu = \nu^\dagger \gamma^0$

So $$\left(U^*_{\alpha k} |\nu_k\rangle\right)^\dagger \gamma^0 = \langle \nu_k | U_{\alpha k} \gamma^0 = \langle \nu_k | \gamma^0 U_{\alpha k} = U_{\alpha k} |\bar\nu_k\rangle$$

Hence $$|\bar\nu_\alpha\rangle = U_{\alpha k} |\nu_k\rangle$$

* I didn't write the sum, was implicit.

b) Derive $P_{\bar\nu_\alpha \to \bar\nu_\beta}$ for antineutrinos.

Let's compute the time evolution for an antineutrino

$$|\bar\nu_\alpha(t)\rangle = \sum_k U_{\alpha k} e^{-iE_k t} |\bar\nu_k\rangle \qquad \text{with} \qquad |\nu_\alpha(0)\rangle = |\nu_\alpha\rangle$$
$$= \sum_\beta \left[\sum_k U_{\alpha k} e^{-iE_k t} U^*_{\beta k} \right] |\bar\nu_\beta\rangle$$

$$P_{\bar\alpha \to \bar\beta}(t) = |\mathcal{A}_{\bar\alpha \to \bar\beta}(t)|^2$$

$$\mathcal{A}_{\bar\alpha \to \bar\beta}(t) \equiv \langle \bar\nu_\beta | \bar\nu_\alpha(t) \rangle = \sum U_{\alpha k} U^*_{\beta k} e^{-iE_k t}$$

Hence $$P_{\bar\alpha \to \bar\beta}(t) = \sum_{k,j} U_{\alpha k} U^*_{\beta k} U^*_{\alpha j} U_{\beta j} \underbrace{e^{-i(E_k - E_j)t}}$$

Using $E_k - E_j \simeq \frac{1}{2E}(m_k^2 - m_j^2)$

$$= \exp\left(-i \frac{\Delta m^2_{kj}}{2E} L\right)$$

c) We want to rewrite $P_{\bar{\alpha}\to\bar{\beta}}(t)$.

Let's recall the result for neutrinos oscillation.

We separated P into a diagonal and off diagonal term, using

$$\sum_{k,j} = \sum_{k=j} + 2\sum_{k>j}$$

After a mathematical tricks we wrote

$$P_{\alpha\to\beta}(L/E) = \delta_{\alpha\beta} - 4\sum_{k>j} Re[U_4]\sin^2\left(\Delta m^2_{kj}\frac{L}{4E}\right) + 2\sum_{k>j} Im[U_4]\sin\left(\Delta m^2_{kj}\frac{L}{2E}\right)$$

To extend this formulas without calculations we can ASSUME CPT invariance:

$$\nu_\alpha \to \nu_\beta \overset{CPT}{\Longrightarrow} \bar{\nu}_\beta \to \bar{\nu}_\alpha$$

and then $P(\nu_\alpha \to \nu_\beta) = P(\bar{\nu}_\beta \to \bar{\nu}_\alpha)$

All we have to do is to change $\alpha \leftrightarrow \beta$.

But $U_4 = U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j} \longrightarrow U_4 = U^*_{\beta k} U_{\alpha k} U_{\beta j} U^*_{\alpha j}$

that is the complex conjugate of the previous term. So all we have to do is mantain the calculation result of neutrinos BUT we must change the sign for the Im{ } part.

$$P_{\bar{\alpha}\to\bar{\beta}}(L/E) = \delta_{\bar{\alpha}\bar{\beta}} - 4\sum_{k>j} Re[U_4]\sin^2\left(\Delta m^2_{kj}\frac{L}{4E}\right) - 2\sum_{k>j} Imm[U_4]\sin\left(\Delta m^2_{kj}\frac{L}{2E}\right)$$

II.4 Oscillation in 2 Flavour case

Starting from the most general 3-Flavour oscillation probability

$$P_{\alpha\to\beta}(L,E) = \sum_{k,j=1}^{3} U_{\alpha k}^{*} U_{\beta k} U_{\alpha j} U_{\beta j}^{*} \exp\left(-i\,\Delta m_{kj}^2 \frac{L}{2E}\right)$$

in the limiting two flavour case $(\alpha,\beta = e,\mu\,;\ k,j = 1,2)$ one has

$$P_{\nu_e\to\nu_\mu}(L,E) = \sin^2 2\theta\ \sin^2 \frac{\Delta m^2 L}{4E}$$

~ Hint: $|\nu_\alpha\rangle = (\,|\nu_e\rangle\ \ |\nu_\mu\rangle\,)^T$ related to $|\nu_i\rangle = (\,|\nu_1\rangle\ \ |\nu_2\rangle\,)^T$ with a 2×2 mixing matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

We have $U = \begin{pmatrix} U_{e1} & U_{e2} \\ U_{\mu 1} & U_{\mu 2} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

and consequently $U^{\dagger} = \begin{pmatrix} U_{e1}^{*} & U_{\mu 1}^{*} \\ U_{e2}^{*} & U_{\mu 2}^{*} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Using the original $P(\nu_\alpha\to\nu_\beta) = \delta_{\alpha\beta} - \sum_{i>j} \mathrm{Re}\{\quad\}\ldots + 2\sum_{i>j} \mathrm{Im}\{\quad\}\ldots$

we have to compute

$$4\,U_{\alpha 2}^{*} U_{\beta 2} U_{\alpha 1} U_{\beta 1}^{*} = -4\sin\theta\cos\theta\cos\theta\sin\theta = -\sin^2 2\theta$$

Then for $\alpha\neq\beta$

$$P(\nu_\alpha\to\nu_\beta) = \delta_{\alpha\beta} - (-\sin^2 2\theta)\sin^2\left(\Delta m_{ij}^2 \frac{L}{4E}\right) + 2\times 0\times \sin\left(\Delta m_{ij}^2 \frac{L}{4E}\right) = \sin^2 2\theta\ \sin^2\left(\Delta m_{ij}^2 \frac{L}{4E}\right)$$

Homework III

III.1

Assume νs are Dirac particles, with a mass arising from Yukawa interaction with Higgs.
After SSB

$$m_f = Y_f^{diag} \frac{v}{\sqrt{2}} \qquad f = (u, d, \ldots, t, e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau)$$

2) Estimate the size of Yuk. coupling of charged fermions.

up: $m_u = Y_u^{diag} \frac{v}{\sqrt{2}}$ since $v \sim 245$ GeV, $m_u \sim 2$ MeV

$$\rightarrow Y_u = \frac{m_u \sqrt{2}}{v} \Rightarrow 0.0075$$

d: $Y_d = 2.64 \times 10^{-5}$ c: $Y_c = 7.39 \times 10^{-3}$ s: $Y_s = 5.5 \times 10^{-4}$

t: $Y_t = 9.93 \times 10^{-1}$ b: $Y_b = 2.40 \times 10^{-2}$

e: $Ye = 2.94 \times 10^{-6}$ μ: $Y_\mu = 6.07 \times 10^{-4}$ τ: $Y_\tau = 1.02 \times 10^{-2}$

b) Heavest neutrino has $m_\nu \sim 0.1$ eV. What's Y_{ν_3}?

$$Y_{\nu_3} \simeq \frac{m_\nu \sqrt{2}}{v} \simeq 10^{-13}$$

Discuss: The Yukawa coupling with a mass of the order of the eV is very small.

III.2 Majorana neutrinos

a) Consider the Field $\psi_R = C\bar{\psi}_L^{\,T}$.
Prove that ψ_R is indeed a righ handed Field.
Hint: also can prove that has no LH component

$$P_L C \bar{\psi}_L^{\,T} = \frac{1}{2}(1-\gamma^5) C \bar{\psi}_L^{\,T} \qquad w/ \quad P_L = \frac{1}{2}(1-\gamma^5) \qquad P_R = \frac{1}{2}(1+\gamma^5)$$

$\psi_R = C\bar{\psi}_L^{\,T} \rightarrow \bar{\psi}_L = (C^{-1}\psi_R)^T \longrightarrow$ Since C is hermitian $C^{-1} = C^T$ and $\gamma^5 C = -C\gamma^5$

$$P_L \psi_R = P_L(C\bar{\psi}_L^{\,T}) = \frac{1}{2}(1-\gamma^5) C (\bar{\psi} P_R)^T = \frac{1}{2}(1-\gamma^5) C P_R^T \bar{\psi}^T = (P_R^T = P_R) = C \underbrace{P_L P_R}_{=0} \bar{\psi}^T = 0$$

Since $\bar{\psi}_L = \bar{\psi} P_R$

b) If ψ is a Majorana Fermion, with a generic charge q, the em current

$j^\mu_{em} = \bar{\psi}\gamma^\mu\psi$ vanishes. $\Longrightarrow$ $q\,\delta_{\mu e} A_\mu = 0$ This implies that the Majorana Fermion cannot couple to A_μ (Photon) $\rightarrow$ Majorana Fermion has no electromagnetic charge.

c) For Majorana neutrinos, we have seen that

$$U = U^{Dirac}(\theta_{12}, \theta_{23}, \theta_{13}, \delta_{CP}) \times \mathrm{diag}(1, e^{i\lambda_2}, e^{i\lambda_3})$$

Derive the expression for Jarlskog invariant, we know that

$$|J_{CP}^\ell| = \mathrm{Im}\left[U_{\mu 3} U_{e2} U_{\mu 2}^* U_{e3}^*\right] \rightarrow \mathrm{Im}\left[U^0_{\mu 3} e^{i\lambda_3} U^0_{e2} e^{i\lambda_2} U^0_{\mu 2} e^{-i\lambda_2} U^0_{e3} e^{-i\lambda_3}\right]$$
$$= \left[U^0_{\mu 3} U^0_{e2} U^{0\,*}_{\mu 2} U^{0\,*}_{e3}\right]$$

This does not depend on the Majorana of Dirac nature of the neutrinos.
During lecture we saw that also we can write

$$J_{CP}^\ell = C_{12} S_{12} C_{23} S_{23} C_{13}^2 S_{13} \sin \delta_{13}$$

III. 3 Oscillation of (Dirac) Majorana neutrinos

Single generation, with Majorana, Dirac mass terms.

$$N_L = \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} \qquad M = \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix} \qquad \text{with} \quad W^T M W = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$$

$$N_L = W n_L = W \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \end{pmatrix}$$

↳ Mass eigenstate

a) Verify that W can be cast as

$$W = \begin{pmatrix} \cos w & \sin w \\ -\sin w & \cos w \end{pmatrix} \times \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{with} \quad \begin{cases} \tan 2w = \dfrac{2 m_D}{m_R - 2\mathrm{Re}(m_L)} \\ \tan 2\alpha = \dfrac{-2\,\mathrm{Im}(m_L)}{\mathrm{Re}(m_L) + m_R - \sqrt{(\mathrm{Re}\,m_L - m_R)^2 + 4m_D^2}} \end{cases}$$

$$W^T M W = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} C_w & -S_w \\ S_w & C_w \end{pmatrix}\begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix}\begin{pmatrix} C_w & S_w \\ -S_w & C_w \end{pmatrix}\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\alpha} C_w & -e^{i\alpha} S_w \\ S_w & C_w \end{pmatrix}\begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix}\begin{pmatrix} C_w & S_w \\ -S_w & C_w \end{pmatrix}\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\alpha}(C_w m_L - S_w m_D) & e^{i\alpha}(C_w m_D - S_w m_R) \\ S_w m_L + C_w m_D & S_w m_D + C_w m_R \end{pmatrix}\begin{pmatrix} C_w & S_w \\ -S_w & C_w \end{pmatrix} \cdot \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\alpha}(c_w^2 m_L - C_w S_w m_D) - e^{i\alpha}(C_w m_D S_w - S_w^2 m_R) & e^{i\alpha}(C_w S_w m_L - S_w^2 m_D) + e^{i\alpha}(c_w^2 m_D - S_w C_w m_R) \\ C_w S_w m_L + C_w^2 m_D - S_w^2 m_D - S_w C_w m_R, & S_w^2 m_L + S_w C_w m_D + C_w S_w m_D + C_w^2 m_R \end{pmatrix} \times \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 1 \end{pmatrix}$$

Hence

$$m_1 = e^{2i\alpha}\left(C_w^2 m_L - C_w S_w m_D - C_w m_D S_w + S_w^2 m_R\right)$$

$$m_2 = S_w^2 m_L + S_w C_w m_D + C_w S_w m_D + C_w^2 m_R$$

$$0 = e^{i\alpha}\left(C_w S_w m_L + C_w^2 m_D - S_w^2 m_D - S_w C_w m_R\right)$$

$$0 = e^{i\alpha}\left(C_w S_w m_L - S_w^2 m_D + C_w^2 m_D - S_w C_w m_R\right)$$

Then

$$0 = (m_L - m_R)\, C_w S_w + m_D (C_w^2 - S_w^2)$$
$$0 = (m_L - m_R)\, C_w S_w + m_D (C_w^2 - S_w^2)$$

$$\Rightarrow \quad 0 = \frac{1}{2}(m_L - m_R)\sin(2w) + m_D \cos(2w)$$

Something is missing ..

$$\frac{\sin(2w)}{\cos(2w)} = \tan(2w) = \frac{-2m_D}{m_L - m_R} = \frac{2m_D}{m_R - m_L}$$

b) To find m_1 and m_2 we look for the eigenstates:

$$\begin{vmatrix} m_L - \lambda & m_D \\ m_D & m_R - \lambda \end{vmatrix} = (m_L - \lambda)(m_R - \lambda) - m^2 = 0$$

$$m_L m_R - \lambda(m_L + m_R) + \lambda^2 - m_D^2 = 0$$

$$\lambda = \frac{(m_L + m_R) \pm \sqrt{(m_L + m_R)^2 - 4\cdot(m_L m_R - m_D^2)}}{2\cdot 1}$$, where we can write

$$m_2^2 = \frac{1}{4}\left[(m_L + m_R + \sqrt{(m_L + m_R)^2 - 4(m_L m_R - m_D^2)}\right]^2$$

$$= \frac{1}{4}\left[(m_L + m_R)^2 + (m_L + m_R)^2 - 4(m_L m_R - m_D)^2 + 2(m_L + m_R)\sqrt{(m_L + m_R)^2 - 4(m_L m_R - m_D^2)}\right] =$$

$$= \frac{1}{4}\left[(m_L + m_R)^2 + (m_L - m_R)^2 + 4m_D^2 + 2(m_L + m_R)\sqrt{(m_L - m_R)^2 + 4m_D^2}\right] =$$

Then

$$\Delta m^2 = m_1^2 - m_2^2 = \frac{4}{4}(m_L + m_R)\sqrt{(m_L - m_R)^2 + 4m_D^2} = \sqrt{(m_L + m_R)^2\left[(m_L - m_R)^2 + 4m_D^2\right]}$$

Homework IV

IV.1 Effective neutrino mass from $0\nu2\beta$ decays

$$U = U^{Dirac}\, \mathrm{diag}\,(1, e^{i\lambda_2}, e^{i\lambda_3})$$

$$\left.\begin{aligned}
U_{e1} &= |U_{e1}^{Dirac}| \\
U_{e2} &= U_{e2}^{Dirac} \cdot e^{i\lambda_2} = |U_{e2}^{Dirac}|\, e^{i\lambda_2} = |U_{e2}|\, e^{i\lambda_2} \\
U_{e3} &= U_{e3}^{Dirac} \cdot e^{i\lambda_3} = |U_{e3}|\, e^{i(\lambda_3 - \delta_{13}^{D})}
\end{aligned}\right\}
\begin{aligned}
\alpha_2 &= 2\lambda_2 \\
\alpha_3 &= 2(\lambda_3 - \delta_{13}^{D})
\end{aligned}$$

$$\Longrightarrow m_{ee} = |U_{e1}|^2 m_1 + |U_{e2}|^2 e^{i2\lambda_2} m_2 + |U_{e3}|^2 e^{i2(\lambda_3 - \delta_{13}^{D})} m_3 = |U_{e1}|^2 m_1 + |U_{e2}|^2 e^{i\alpha_2} m_2 + |U_{e3}|^2 e^{i\alpha_3} m_3$$

Also

$$m_2 = \sqrt{m_1^2 + \Delta m_\odot^2} \underset{m_1^2 \ll \Delta m_\odot^2}{\simeq} \sqrt{\Delta m_\odot}$$

$$m_3 = \sqrt{m_1^2 + \Delta m_\oplus^2} \underset{m_1^2 \ll \Delta m_\oplus^2}{\simeq} \sqrt{\Delta m_\oplus^2}$$

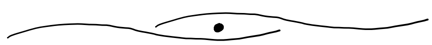
a) I.O. $\quad m_3,\ m_1^2 \simeq m_3^2 + \Delta m_\oplus^2 \qquad m_2^2 \simeq m_3^2 + \Delta m_\oplus^2 + \Delta m_\odot^2$

$$m_{ee} = |U_{e1}|^2 m_1 + |U_{e2}|^2 e^{i\alpha_2} m_2 + |U_{e3}|^2 e^{i\alpha_3} m_3$$

$$= |U_{e1}|^2 \sqrt{m_3^2 + \Delta m_\oplus^2} + |U_{e2}|^2 e^{i\alpha_2} \sqrt{m_3 + \Delta m_\oplus^2 + \Delta m_\odot^2} + |U_{e3}|^2 m_3 e^{i\alpha_3}$$

Since we can neglect m_3^2 and $\Delta m_\odot^2$ w.r.t $\Delta m_\oplus^2$ we will have

$$|m_{ee}| = \left| |U_{e1}|^2 \sqrt{\Delta m_\oplus^2} + |U_{e2}|^2 e^{i\alpha_2} \sqrt{\Delta m_\oplus^2} + |U_{e3}|^2 e^{i\alpha_3} m_3 \right|$$



b) Following the notes, taking the limit $|U_{e3}| \sim 0$

$$|m_{ee}| = \left| |U_{e2}|^2 \sqrt{\Delta m^2_\odot} + |U_{e2}|^2 e^{i\alpha_2} \sqrt{\Delta m^2_\odot} \right|$$

which can provide the bounds

$$|m_{ee}| \le \left| |U_{e2}|^2 \sqrt{\Delta m^2_\odot} + |U_{e2}|^2 \sqrt{\Delta m^2_\odot} \right| \quad \text{for} \quad e^{i\alpha_2} = 1 \qquad \text{Upper bound}$$

$$|m_{ee}| \ge \left| |U_{e2}|^2 \sqrt{\Delta m^2_\odot} - |U_{e2}|^2 \sqrt{\Delta m^2_\odot} \right| \quad \text{for} \quad e^{i\alpha_2} = -1 \qquad \text{Lower bound}$$

c) Using numbers for N.O and I.O

NO $\quad |m_{ee}| \le 1,743$ meV

IO $\quad 18,246$ MeV $\le |m_{ee}| \le 48,39$ meV

$\Rightarrow$ The ranges are different and not compatible.

With high sensitivity and precision we should be able to determine and distinguish in which order neutrino are.

ACHTUNG: We've neglected the mass of the lightest neutrino and $\Delta^2 m_e \gg \Delta m^2_\odot$, m_2^2 can also be neglected w.r.t. $\Delta m^2_\odot$ in N.O.

Homework V

V.1

a) 2 species of sterile fermions $L(\nu_R) = L(\chi) = 1$

$$\mathcal{L}_{MASS} = \underbrace{-\frac{1}{2} m_R \nu_R^{+} C^{+} \nu_R - \frac{1}{2} m_\chi \bar{\chi}^{c} \chi}_{\text{violate Lepton Number}} - y^{\nu} \bar{\nu}_R \tilde{\varphi}^{+} \nu_L - M_R \bar{\nu}_R \chi - M_L \bar{\nu}_L \chi + h.c.$$

(No $-\frac{1}{2} m_L \nu_L^{+} C^{+} \nu_L$) because violates $SU_L(2) \times U(1)$ invariance

b) Subset of terms

$$\mathcal{L}_{MASS} = -y^{\nu} \bar{\nu}_R \tilde{\varphi}^{*} \nu_L - M_R \bar{\nu}_R \chi - \frac{1}{2} M_\chi \bar{\chi}^{c} \chi + h.c.$$

Electroweak symmetry breaking

$$m_D = y_\nu \frac{v}{\sqrt{2}} \qquad \text{so} \qquad \mathcal{L}_{MASS} = -m_D \bar{\nu}_R \nu_L - M_R \bar{\nu}_R \chi - \frac{1}{2} M_\chi \bar{\chi}^{c} \chi$$

Now $n = \begin{pmatrix} \nu_L \\ \nu_R^c \\ \ell \end{pmatrix}$ and let's write the Lagrangian of the masses term

$$\mathcal{L}_{MASS} = \frac{1}{2} n^{+} C^{+} M n + h.c.$$

Consequence $M_{11} = M_{22} = 0 \gg$ Majorana mass ν_L and ν_R which don't appear in our Lagrangian.

For the same reason $M_{13} = M_{31} = 0$ since the $\mathcal{L}$ doesn't contain the cross terms ν_L / χ.

Now starting from $\psi^c = C \bar{\psi}^T$

$$\bar{\psi}^{c} = \overline{C \bar{\chi}^{t}} = \gamma^0 (C \bar{\chi}^T)^{\dagger} = \gamma^0 (\bar{\chi}^{T\dagger} C^{+}) = \gamma^0 (\gamma^0 \chi^{+\dagger} C^{+})$$

$$= \gamma^0 \gamma^0 \chi^{+} C^{+} = \chi^{+} C^{+} \qquad \Big\} \; \psi^c = \chi^T C^{+} \wedge \bar{\psi} = \chi^{c^T} C^{+}$$

The matrix is $\mathcal{M} = \begin{pmatrix} 0 & m_D & 0 \\ m_D & 0 & M_R \\ 0 & M_R & M_\chi \end{pmatrix}$ $\longrightarrow$ To prove it let's compute $\frac{1}{2} n^{+} C^{+} M n$ and find back our $\mathcal{L}$ original.

$$\mathcal{L} = \frac{1}{2} n^T C^\dagger M n = \frac{1}{2} (\nu_L^T, \nu_R^{cT}, \chi^T) C^\dagger \begin{pmatrix} m_D \nu_R^c \\ m_D \nu_L + M_R \chi \\ M_R \nu_R^c + M_\chi \chi \end{pmatrix}$$

$$= \frac{1}{2} \nu_L^T \cancel{C^\dagger} m_D \cancel{C} \bar{\nu}_R^T + \frac{1}{2} \bar{\nu}_R m_D \nu_L + \frac{1}{2} \bar{\nu}_R M_R \chi + \frac{1}{2} \chi^T \cancel{C^\dagger} M_R \cancel{C} \bar{\nu}_R^T + \frac{1}{2} \bar{\chi}^c M_\chi \chi$$

$$= \frac{1}{2} m_D \bar{\nu}_R \nu_L + \frac{1}{2} m_D \bar{\nu}_R \nu_L + \frac{1}{2} M_R \bar{\nu}_R \chi + \frac{1}{2} M_D \bar{\nu}_R \chi + \frac{1}{2} M_\chi \bar{\chi}^c \chi$$

$$= m_D \bar{\nu}_R \nu_L + M_R \bar{\nu}_R \chi + \frac{1}{2} M_\chi \bar{\chi}^c \chi$$

c) Let's consider the limit $M_\chi \ll m_D \ll M_R$

Now, the eigenvalues coming from $\det(M - \lambda \mathbb{1}) = 0$

$$\det \begin{pmatrix} -\lambda & m_D & 0 \\ m_D & -\lambda & M_R \\ 0 & M_R & M_\chi - \lambda \end{pmatrix} = -\lambda [-\lambda (M_\chi - \lambda) - M_R^2] - m_D [m_D (M_\chi - \lambda) - 0] = 0$$

$$\lambda^2 (M_\chi - \lambda) + \lambda M_R^2 - m_D^2 (M_\chi - \lambda) = 0$$

$$-\lambda^3 + \lambda^2 M_\chi + \lambda (M_R^2 + m_D^2) - m_D^2 M_\chi = 0$$

We expect 1 small eigenvalues related to the small coupling term and 2 big one

Hence we can use the approximate λ^2 and $\lambda^3 \approx 0$ to find the small eigenvalue.

$$\lambda_0 \simeq \frac{m_D^2 M_\chi}{M_R^2 + m_D^2}$$

Then I can write the equation as $(A\lambda^2 + B\lambda + C) \left(\lambda - \overbrace{\frac{m_D^2 M_\chi}{M_R^2 + m_D^2}}^{K} \right) = 0$

$$A\lambda^3 + B\lambda^2 + C\lambda - KA\lambda^2 - BK\lambda - CK = 0 \Rightarrow A\lambda^3 + (B - KA)\lambda^2 + (C - BK)\lambda - CK = 0$$

By comparing the coefficients we can get $A = -1$ $\quad B = M_x - \frac{m_D^2 M_x}{M_R^2 + m_D^2}$

Ignoring terms with M_x^2 $\quad C = M_R^2 + m_D^2 + \frac{m_D^2 M_x^2}{M_R^2 + m_D^2} - \frac{m_D^4 M_x^2}{(M_R^2 + m_D^2)^2}$

$$-\lambda^2 + \left(M_x - \frac{m_D^2 M_x}{M_R^2 + m_D^2} \right) \lambda + M_R^2 + m_D^2 + \overbrace{\frac{m_D^2 M_x^2}{M_R^2 + m_D^2} - \frac{m_D^4 M_x^2}{(M_R^2 + m_D^2)^2}}^{\simeq 0} = 0$$

$$\lambda_{1,2} \simeq \left(\frac{m_D^2 M_x}{M_R^2 + m_D^2} - M_x \pm \sqrt{ \underbrace{\left(M_x - \frac{m_D^2 M_x}{M_R^2 + m_D^2} \right)^2}_{\simeq 0} + 4 (M_R^2 + m_D^2) } \right) \cdot -\frac{1}{2}$$

$$\simeq \sqrt{(M_R^2 + m_D^2)} + \frac{M_x}{2} \left(1 - \frac{m_D^2}{M_R^2 + m_D^2} \right) \xrightarrow[M_x \ll m_D]{m_D \ll M_R} \simeq \pm \sqrt{M_R^2} = \pm M_R$$

$$\begin{cases} \lambda_0 \simeq \frac{m_D^2 M_x}{M_R^2} = \frac{v^2}{2} \frac{Y_L^2}{M_R^2} M_x \\ \lambda_{1,2} \simeq \pm M_R \end{cases}$$

$M_{\nu_{R2}} = M_{\nu_{R2}} \simeq M_R$ if we keep only the physical solution

d) We know $v \simeq 10^{11}$ eV so we have $m_\nu \simeq 10^{22} \frac{Y_L^2}{M_R^2} M_x$. Since extremely small values of a coupling constant M_R^2 are very difficult to explain theoretically, we need a very large value M_R.
This would correspond to a seesaw type 1 like $\Rightarrow$ the mediator of high mass that suppress the Yukawa couplings of neutrinos is a RH Fermion

e) If the heavy states had the exact solution $M_{\nu_{R2}} = M_{\nu_{R2}} = M_R$ this would correspond to the Dirac case.
In reality we have it as an approximation so that 2 states are massive Major. neutrinos which are almost degenerate and of opposite CP parities. They are difficult to disentangle from a single Dirac neutrino so their behaviour is called pseudo-Dirac.