Exercise 2 | Pag 48

Check if

$$\varphi(x) = \varphi_0(x) + \int d^4y\, G(x,y) J(y) - \frac{g}{3!}\int d^4y\, G(x,y)\, \varphi^3(y)$$

is solution for $\boxed{(\square_x + m^2)\varphi = J - \frac{g}{3!}\varphi^3}$ with $G(x) = i\int d\hat{p}\, (e^{-ipx} - e^{ipx})$

$$(\square + m^2)\left[\varphi_0(x) + \int d^4y\, G(x,y) J(y) - \frac{g}{3!}\int d^4y\, G(x,y)\varphi^3(y)\right] =$$

$$= m^2\left(\varphi_0(x) + \int d^4y\, G(x,y)J(y) - \frac{g}{3!}\int d^4y\, G(x,y)\varphi^3(y)\right) + \square\varphi_0(x) + \int d^4y\, \square G(x,y) J(y) - \frac{g}{3!}\int d^4y\, \square G(x,y)\varphi^3(y)$$

$$= (m^2 + \square)\varphi_0(x) + \int d^4y\, (\square + m^2) G(x,y) J(y) - \frac{g}{3!}\int d^4y\, (m^2 + \square) G(x,y)\varphi^3(y)$$

but we know $(\square + m^2) G(x,y) = \delta(x-y)$

$$= \underbrace{(m^2 + \square)\varphi_0(x)}_{=0} + \int d^4y\, \delta(x-y) J(y) - \frac{g}{3!}\int d^4y\, \delta(x-y)\varphi^3(y) = 0 + J(x) - \frac{g}{3!}\varphi^3(x)$$

solution of homogeneous

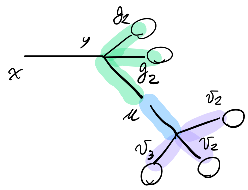
$\square$

Exercise 3 | Pag 51

Compute the order g^3

Let's start from the g^2 term

$$+3\left(-\frac{g}{3!}\right)\left(-\frac{g}{3!}\right)\int d^4y\, G(x,y)\left[\int d^4z\, G(y,z)\, J(z)\right]^2 \int d^4\mu\, G(y,\mu) \times \left[\int d^4v\, G(\mu,v)\, J(v)\right]^3$$



$$\times\left[\int d^4v_1\, G(\mu,v_1)\, J(v_1)\cdot\int d^4v_2\, G(\mu,v_2)\, J(v_2)\int d^4v_3\, G(\mu,v_3)\, J(v_3)\right]$$

To build the g^3 term let's remind the expansion $(\alpha - g\beta)^3 = \alpha^3 - 3g\alpha^2\beta + 3g^2\alpha\beta^2 - g^3\beta^3$
The term that we have to expand is $+3g^2\alpha\beta^2$

$\alpha = \int d^4z\, G(y,z)\, J(z)$ | $\beta = \int d^4\mu\, G(y,\mu)\, \varphi^3(\mu)$ | $g = -\frac{g}{3!}$

Remind that all the expansion was multiplied by: $-\frac{g}{3!}\int d^4y\, G(x,y)\left[\dots + 3g^2\alpha\beta^2\right]$

Hence

$$\Rightarrow -\frac{g}{3!}\int d^4y\, G(x,y)\left[+3\left(-\frac{g}{3!}\right)\left(-\frac{g}{3!}\right)\int d^4z\, G(y,z)\, J(z)\left(\int d^4\mu\, G(y,\mu)\, \varphi^3(\mu)\right)^2\right]$$

g^3 Term

$$\left[\int d^4v\, G(\mu,v)\, J(v) - \cancel{\frac{g}{3!}\int d^4v\, G(\mu,v)\, \varphi^3(v)}\right]^3$$

I want g^3 power

$$\Rightarrow -\frac{g}{3!}\int d^4y\, G(x,y)\left[+3\left(-\frac{g}{3!}\right)\left(-\frac{g}{3!}\right)\int d^4z\, G(y,z)\, J(z)\left(\int d^4\mu\, G(y,\mu)\left[\int d^4v\, G(\mu,v)\, J(v)\right]^3\right)^2\right]$$

$$\left(\int d^4\mu\, G(y,\mu)\left[\int d^4v\, G(\mu,v)\, J(v)\right]^3\right)^2 =$$

$$\left(\int d^4\mu_1\, G(y,\mu_1)\underbrace{\left[\int d^4v\, G(\mu_1,v)\, J(v)\right]^3}\right) \times \left(\int d^4\mu_2\, G(y,\mu_2)\underbrace{\left[\int d^4v\, G(\mu_2,v)\, J(v)\right]^3}\right)$$

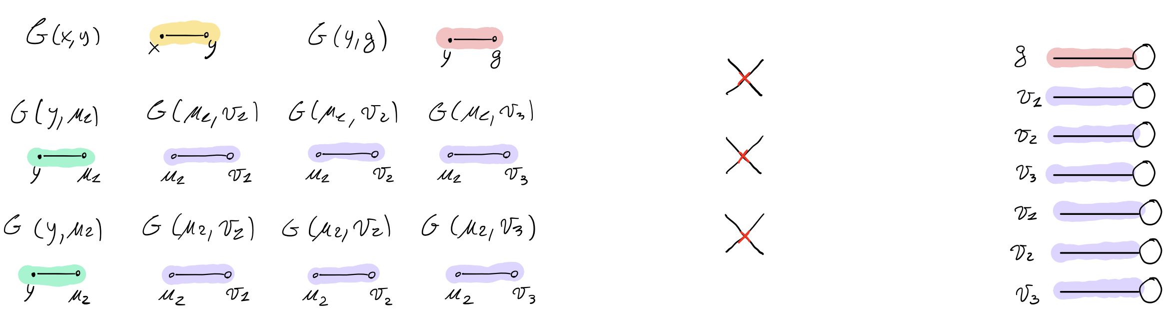
$$\left[\int d^4v_1\, G(\mu_1,v_1) J(v_1) \cdot \int d^4v_2\, G(\mu_1,v_2) J(v_2) \int d^4v_3\, G(\mu_1,v_3) J(v_3)\right] \qquad \left[\int d^4v_1\, G(\mu_2,v_1) J(v_1) \cdot \int d^4v_2\, G(\mu_2,v_2) J(v_2) \int d^4v_3\, G(\mu_2,v_3) J(v_3)\right]$$

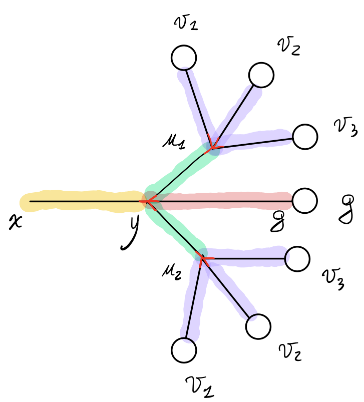
In the end

$$-\frac{g}{3!}\int d^4y\, G(x,y)\left[+3\left(-\frac{g}{3!}\right)\left(\frac{-g}{3!}\right)\int d^4g\, G(y,g)\, J(g)\left\{\int d^4\mu_1\, G(y,\mu_1)\left[\int d^4v_1\, G(\mu_1,v_1) J(v_1) \cdot \int d^4v_2\, G(\mu_1,v_2) J(v_2) \int d^4v_3\, G(\mu_1,v_3) J(v_3)\right] \times \right.\right.$$

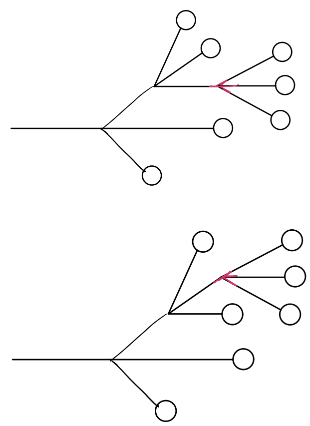
$$\left.\left.\int d^4\mu_2\, G(y,\mu_2)\left[\int d^4v_1\, G(\mu_2,v_1) J(v_1) \cdot \int d^4v_2\, G(\mu_2,v_2) J(v_2) \int d^4v_3\, G(\mu_2,v_3) J(v_3)\right]\right\}\right]$$

Green Functions Vertex J terms





But also ...



etc

We should take into account symmetry factor

Exercise 4 | Pag 52

The action

$$S[\varphi,\varphi^*] = S_1[\varphi_1] + S_2[\varphi_2]$$

Show that $S[\varphi,\varphi^*] = \int d^4x \left(\partial_\mu\varphi^*\partial^\mu\varphi - m^2\varphi^*\varphi\right)$

We know that
$$\begin{cases} \varphi(x) = \frac{1}{\sqrt{2}}\left(\varphi_1(x) + i\,\varphi_2(x)\right) \\ \varphi^*(x) = \frac{1}{\sqrt{2}}\left(\varphi_1(x) - i\,\varphi_2(x)\right) \end{cases}$$

$$\Rightarrow \int d^4x \left(\partial_\mu\varphi^*\partial^\mu\varphi - m^2\varphi^*\varphi\right) =$$

$$= \int d^4x \left(\partial_\mu\left(\frac{1}{\sqrt{2}}\varphi_1 - i\varphi_2\right)\partial^\mu\frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2) - m^2\left(\frac{1}{2}\varphi_1^2 + \varphi_2^2\right)\right.$$

$$= \int d^4x \left(\frac{1}{\sqrt{2}}\partial_\mu\varphi_1 - \frac{i}{\sqrt{2}}\partial_\mu\varphi_2\right)\left(\frac{1}{\sqrt{2}}\partial^\mu\varphi_1 + \frac{i}{\sqrt{2}}\partial^\mu\varphi_2\right) - \frac{m^2}{2}\left(\varphi_1^2 + \varphi_2^2\right)$$

$$= \int d^4x\, \underline{\frac{1}{2}\partial_\mu\varphi_1\partial^\mu\varphi_1} \cancel{- \frac{i}{2}\partial_\mu\varphi_2\partial^\mu\varphi_1} + \underline{\frac{1}{2}\partial_\mu\varphi_2\partial^\mu\varphi_2} \cancel{+ \frac{i}{2}\partial_\mu\varphi_1\partial^\mu\varphi_2} - \underline{\frac{m^2}{2}}\left(\underline{\varphi_1^2 + \varphi_2^2}\right)$$

$$= S_1[\varphi_1] + S_2[\varphi_2]$$

because

- $\partial_\mu\varphi_1\partial^\mu\varphi_2 = \partial_\mu\varphi_2\partial^\mu\varphi_1$
- $\partial_\mu\varphi_1 g^{\mu\nu}\partial_\nu\varphi_2 = \partial_\mu\varphi_2 g^{\mu\nu}\partial_\nu\varphi_1$
- $\underline{\partial_\mu\varphi_2}\, g^{\mu\nu}\, \underline{\partial_\nu\varphi_1} = \underline{\partial_\nu\varphi_2}\, g^{\mu\nu}\, \underline{\partial_\mu\varphi_1}$

Equation of Motion

$$\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = 0 \qquad\qquad \frac{\partial\mathcal{L}}{\partial\varphi^*} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^*)} = 0$$

- $\frac{\partial\mathcal{L}}{\partial\varphi} = -m^2\varphi^*$ $\qquad$ • $\frac{\partial\mathcal{L}}{\partial\varphi^*} = -m^2\varphi$

$\left[\partial_\mu\varphi = \partial^\nu\varphi\, g_{\nu\mu}\right.$

$\left.\partial^\mu\varphi = \partial_\nu\varphi\, g^{\nu\mu}\right]$

- $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = \partial^\mu\varphi^*$ $\qquad$ • $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^*)} = \partial^\mu\varphi \;\rightarrow\; \partial_\mu\partial^\mu\varphi$

$$\Rightarrow (\Box + m^2)\varphi^* = 0 \qquad\qquad \Rightarrow (\Box + m^2)\varphi = 0$$

Exercise 5

Express j^μ and Q in terms of a and b functions

$$\boxed{j^\mu = i\,[\varphi^* \partial^\mu \varphi - \varphi \partial^\mu \varphi^*]} \qquad \boxed{Q = \int d^3\vec{x}\; j^0(\vec{x})}$$

with

$$\varphi(x) = \int d\hat{p}\,\left[e^{-ipx} a(\vec{p}) + e^{ipx} b^*(\vec{p})\right]$$

$$\varphi^*(x) = \int d\hat{p}\,\left[e^{-ipx} b(\vec{p}) + e^{ipx} a^*(\vec{p})\right]$$

Let's compute $\partial^\mu \varphi$

$$\partial^\mu \varphi = \int d\hat{p}\; \partial^\mu [e^{-ipx} a(\vec{p})] + \partial^\mu [e^{+ipx} b^*(\vec{p})]$$

$$= \int d\hat{p}\; a(p)\, \partial^\mu e^{-ipx} + b^*(p)\, \partial^\mu e^{+ipx}$$

$$= \int d\hat{p}\; a(p)\,(-ip^\mu)\, e^{-ipx} + b^*(p)\,(ip^\mu)\, e^{+ipx}$$

and so $\partial^\mu \varphi^*$

$$\partial^\mu \varphi^* = \int d\hat{p}\; \partial^\mu [e^{+ipx} a^*(\vec{p})] + \partial^\mu [e^{-ipx} b(\vec{p})]$$

$$= \int d\hat{p}\; a^*(p)\, \partial^\mu e^{+ipx} + b(p)\, \partial^\mu e^{-ipx}$$

$$= \int d\hat{p}\; a^*(p)\,(+ip^\mu)\, e^{+ipx} + b(p)(-ip^\mu)\, e^{-ipx}$$

So $\quad \varphi^* \partial^\mu \varphi = \int d\tilde{p} \int d\tilde{k} \, [e^{-ipx} b(\vec{p}) + e^{ipx} a^*(\vec{p})][a(k)(-ik^\mu) e^{-ikx} + b^*(k)(ik^\mu) e^{+ikx}]$

when you multiply 2 fields, always ⚠ use 2 different integration variables

$$= \int d\tilde{p} \int d\tilde{k} \, [e^{-ipx} e^{-ikx} b(p) a(k)(-ik^\mu)] + [e^{-ipx} b(p) b^*(k)(ik^\mu) e^{+ikx}] +$$

$$[e^{ipx} a^*(p) a(k)(-ik^\mu) e^{-ikx}] + [e^{ipx} a^*(p) b^*(k)(ik^\mu) e^{+ikx}]$$

And $\quad \varphi \partial^\mu \varphi^* = \int d\tilde{p} \int d\tilde{k} \, [e^{-ipx} a(\vec{p}) + e^{ipx} b^*(\vec{p})][a^*(k)(+ik^\mu) e^{+ikx} + b(k)(-ik^\mu) e^{-ikx}]$

$$= \int d\tilde{p} \int d\tilde{k} \, [e^{-ipx} e^{+ikx} a(p) a^*(k)(+ik^\mu)] + [e^{-ipx} a(p) b(k)(-ik^\mu) e^{-ikx}] +$$

$$[e^{ipx} b^*(p) a^*(k)(ik^\mu) e^{+ikx}] + [e^{ipx} b^*(p) b(k)(-ik^\mu) e^{-ikx}]$$

I think that the exercise stops here because to do semplifications we need commutation relations... but we didn't QUANTIZE THE FIELDS yet!

The charge Q

$$Q = \int d^3\vec{x} \, j^0(\vec{x}) = \int d^3\vec{x} \, i \, [\varphi^* \partial^0 \varphi - \varphi \partial^0 \varphi^*]$$

- $\varphi^* \partial^0 \varphi = \int d\tilde{p} \int d\tilde{k} \, [e^{-ipx} e^{-ikx} \cancel{b(p) a(k)} (-i\omega_k)] + [e^{-ipx} b(p) b^*(k)(i\omega_k) e^{+ikx}] +$

$$[e^{ipx} a^*(p) a(k)(-i\omega_k) e^{-ikx}] + [e^{ipx} a^*(p) \cancel{b^*(k)} (i\omega_k) e^{+ikx}]$$

- $\varphi \partial^0 \varphi^* = \int d\tilde{p} \int d\tilde{k} \, [e^{-ipx} e^{+ikx} a(p) a^*(k)(+i\omega_k)] + [e^{-ipx} a(p) \cancel{b(k)} (-i\omega_k) e^{-ikx}] +$

$$[e^{ipx} b^*(p) \cancel{a^*(k)} (i\omega_k) e^{+ikx}] + [e^{ipx} b^*(p) b(k)(-i\omega_k) e^{-ikx}]$$

$$Q=\int d^3\vec{x}\, j^0(\vec{x}) = \int d^3\vec{x}\, i\left[\varphi^*\partial^0\varphi - \varphi\partial^0\varphi^*\right]$$

$$= \int d^3\vec{x}\, i\Big\{\int d\tilde{p}\int d\tilde{k}\, \left[e^{-ipx} b(p)\, b^*(k)\,(i\omega_k)\, e^{+ikx}\right] + \left[e^{ipx} a^*(p)\, a(k)\,(-i\omega_k)\, e^{-ikx}\right] -$$

$$\Big[\int d\tilde{p}\int d\tilde{k}\, \left[e^{-ipx} e^{+ikx} a(p)\, a^*(k)\,(+i\omega_k)\right] + \left[e^{ipx} b^*(p)\, b(k)\,(-i\omega_k)\, e^{-ikx}\right]\Big\}$$

$k \leftrightarrow p$

$$\int d\tilde{k}\int d\tilde{p}\, \left[e^{-ikx} e^{+ipx} a(k)\, a^*(p)\,(+i\omega_p)\right] + \left[e^{ikx} b^*(k)\, b(p)\,(-i\omega_p)\, e^{-ipx}\right]\Big\}$$

And so $Q = \int d^3\vec{x}\, i\left\{\int d\tilde{p}\int d\tilde{k}\, \left[e^{-ipx} b(p)\, b^*(k)\,(+i\omega_k + i\omega_p)\, e^{+ikx}\right] + \left[e^{ipx} a^*(p)\, a(k)\,(-i\omega_k - i\omega_p)\, e^{-ikx}\right]\right\}$

Now $(2\pi)^3\delta(\vec{p}-\vec{k}) = \int d^3x\, e^{-i\vec{x}(\vec{p}-\vec{k})}$ $\qquad (2\pi)^3\delta(-\vec{p}+\vec{k}) = \int d^3x\, e^{i\vec{x}(\vec{p}-\vec{k})}$

$$Q = \int d^3\vec{x}\, i \int\frac{d\vec{p}}{(2\pi)^3 2\omega_p}\int\frac{d\vec{k}}{(2\pi)^3 2\omega_k}\, e^{-i\omega_p x_0}\, e^{i\omega_k x_0}\left\{\left[e^{+i\vec{p}\vec{x}} b(p)\, b^*(k)\,(+i\omega_k + i\omega_p)\, e^{-i\vec{k}\vec{x}}\right] + \left[e^{i\vec{p}\vec{x}} a^*(p)\, a(k)\,(-i\omega_k - i\omega_p)\, e^{-i\vec{k}\vec{x}}\right]\right\}$$

$$= \int d\vec{p}\, \frac{1}{\cancel{(2\pi)^3}}\frac{1}{(2\pi)^3}\frac{1}{2\omega_p}\frac{1}{2\omega_k}\, e^{-i\omega_p x_0}\, e^{i\omega_k x_0}\, i\left\{\cancel{(2\pi)^3}\,\underbrace{\delta(-\vec{p}+\vec{k})}_{\delta(\vec{p}-\vec{k})}\,(-i\omega_k - i\omega_p)\left(-b(p)\, b^*(k) + a^*(p)\, a(k)\right)\right\}$$

Future Fabio: was here the "-"

$$= \int d\vec{p}\, \frac{1}{(2\pi)^3}\frac{1}{2\omega_p}\frac{1}{\cancel{2\omega_p}}\left\{\cancel{(2\omega_p)}\left[-b(p)\, b^*(p) + a^*(p)\, a(p)\right]\right\}$$

$$= \int d\tilde{p}\, \left[-b(p)\, b^*(p) + a^*(p)\, a(p)\right]$$

For sure there something wrong, but you got the idea

Exercise 5.1. Fix it

Future Fabio: done

Exercise 6 | QFT 7

Let's consider

$$\begin{cases} \varphi(x) \longrightarrow \varphi'(x) = e^{i\alpha}\varphi(x) \\ \varphi^*(x) \longrightarrow \varphi^{*\prime}(x) = e^{-i\alpha}\varphi^*(x) \end{cases} \qquad \alpha \in \mathbb{R} \qquad \text{with } \mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi$$

Exercise: ◦ $\mathcal{L}$ has the same structure before and after

◦ $S'[\varphi', \varphi'^*] = S[\varphi, \varphi^*]$

$$\mathcal{L}' = \partial_\mu (e^{-i\alpha}\varphi^*)\, \partial^\mu (e^{i\alpha}\varphi) - m^2 (e^{-i\alpha} e^{i\alpha} \varphi^* \varphi)$$

$$\overset{!}{=} \cancel{e^{-i\alpha}}\, \cancel{e^{i\alpha}}\, \partial_\mu \varphi^* \partial^\mu \varphi - m^2 \cancel{e^{i\alpha}}\, \cancel{e^{-i\alpha}}\, \varphi^* \varphi$$

Exercise 7 | QFT 8

$$\left| \begin{array}{l} \psi(\vec{x}) = \sum a_i \langle \vec{x} | U_i \rangle \\ \psi^+(\vec{x}) = \sum a_i^+ \langle U_i | \vec{x} \rangle \end{array} \right. \quad \xrightarrow[\text{to obtain}]{\text{invert them}} \quad \begin{array}{l} a_i = \int d^3\vec{x}\, \psi(\vec{x}) \langle U_i | \vec{x} \rangle \\ a_i^+ = \int d^3\vec{x}\, \psi^+(\vec{x}) \langle \vec{x} | U_i \rangle \end{array}$$

---

1)

$$\psi(\vec{x}) = \sum a_i \langle \vec{x} | U_i \rangle$$

$$\psi(\vec{x}) \langle U_j | \vec{x} \rangle = \sum a_i \langle \vec{x} | U_i \rangle \langle U_j | \vec{x} \rangle$$

$$\int d^3x\, \psi(\vec{x}) \langle U_j | \vec{x} \rangle = \int d^3x \sum a_i \langle \vec{x} | U_i \rangle \langle U_j | \vec{x} \rangle$$

$$\overset{|}{=} \sum a_i \int d^3x\, \langle \vec{x} | U_i \rangle \langle U_j | \vec{x} \rangle$$

$$\overset{!}{=} \sum a_i\, \delta_{ij} = a_j$$

$$\int d^3x\, \underbrace{\langle x | U_i \rangle}_{U_i(x)} \underbrace{\langle U_j | \vec{x} \rangle}_{U_j^*(x)} = \delta_{ij}$$

2)

$$\psi^\dagger(\vec{x}) = \sum a_i^\dagger \langle U_i | \vec{x} \rangle$$

$$\psi^\dagger(\vec{x}) \langle \vec{x} | U_j \rangle = \sum a_i^\dagger \langle \vec{x} | U_i \rangle^* \langle U_j | \vec{x} \rangle^*$$

$$\int d^3x \, \psi^\dagger(\vec{x}) \langle \vec{x} | U_j \rangle = \int d^3x \sum a_i^\dagger \langle \vec{x} | U_i \rangle^* \langle U_j | \vec{x} \rangle^*$$

$$= \sum a_i^\dagger \int d^3x \, \langle \vec{x} | U_i \rangle^* \langle U_j | \vec{x} \rangle^* = a_j^\dagger$$

$$\int d^3x \, \underbrace{\langle \vec{x} | U_i \rangle^*}_{U_i^*(x)} \underbrace{\langle U_j | \vec{x} \rangle^*}_{U_j(x)} = \delta_{ij}$$

Exercise 8 | QFT 8

Prove $[\psi(x), \psi(y)] = 0$

$$\psi(x) = \sum_i a_i \langle \vec{x} | U_i \rangle$$

$$\psi(y) = \sum_i a_i \langle \vec{y} | U_i \rangle$$

$$[\psi(x), \psi(y)] = \psi(x)\psi(y) - \psi(y)\psi(x)$$

$$= \cancel{\sum_i \sum_j a_i a_j \langle \vec{x} | U_i \rangle \langle \vec{y} | U_j \rangle - \sum_j \sum_i a_j a_i \langle \vec{y} | U_i \rangle \langle \vec{x} | U_j \rangle}$$

numbers

$$[\psi^\dagger(x), \psi^\dagger(y)] = \text{Same}$$

$$[\psi(\vec{x}), \psi^\dagger(\vec{y})] = \sum_i a_i \langle \vec{x} | U_i \rangle \sum_j a_j^\dagger \langle U_j | \vec{y} \rangle - \sum_j a_j^\dagger \langle U_j | \vec{y} \rangle \sum_i a_i \langle \vec{x} | U_i \rangle$$

$$= \sum_i \sum_j a_i a_j^\dagger \langle \vec{x} | U_i \rangle \langle U_j | \vec{y} \rangle - \sum_j \sum_i a_j^\dagger a_i \langle U_j | \vec{y} \rangle \langle \vec{x} | U_i \rangle$$

$$a_i a_j^\dagger - a_j^\dagger a_i = \delta_{ij}$$

$$= \sum_{ij} \delta_{ij} \langle \vec{x} | \vec{y} \rangle = \delta(\vec{x} - \vec{y})$$

Prove that $N = \sum_i a_i^\dagger a_i$

$$\sum_i a_i^\dagger a_i = \int d^3x\ \psi^\dagger(\vec{x}) \langle \vec{x} | U_i \rangle \int d^3y\ \psi(y) \langle U_i | \vec{y} \rangle$$

$$= \int d^3x\ \psi^\dagger(\vec{x}) \int d^3x\ \psi(x) \langle U_i | \vec{x} \rangle \langle \vec{x} | V_i \rangle$$

$$= \int d^3x\ \psi^\dagger(x)\ \psi(x)$$

$\varphi \neq \vec{x} = 0$

using $\int d^3x\ |x\rangle\langle x| = 1$

using $\langle U_i | U_N \rangle = 1$

Exercise 9 | QFT 8

We define $a(\vec{p})$ and $a^\dagger(\vec{p})$ by

$$\psi(\vec{x}) = \int \frac{d\vec{p}}{(2\pi)^3}\ e^{i\vec{p}\cdot\vec{x}} a(\vec{p}) \qquad \psi^\dagger(\vec{x}) = \int \frac{d\vec{p}}{(2\pi)^3}\ e^{-i\vec{p}\cdot\vec{x}} a^\dagger(\vec{p})$$

Those can be inverted

$$a(\vec{p}) = \int d^3\vec{x}\ e^{-i\vec{p}\cdot\vec{x}}\ \psi(\vec{x}) \qquad a^\dagger(\vec{p}) = \int d^3\vec{x}\ e^{i\vec{p}\cdot\vec{x}}\ \psi^\dagger(\vec{x})$$

$\underline{1|}$ $$\psi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3}\ e^{i\vec{p}\cdot\vec{x}}\ a(p) \qquad (*)$$

Let's integrate both sides by $\vec{x}$

$$\int d^3\vec{x}\ \psi(\vec{x}) = \int d^3x \int \frac{d^3\vec{p}}{(2\pi)^3}\ e^{+i\vec{p}\cdot\vec{x}}\ a(\vec{p})$$

$\cdot\ e^{-i\vec{k}x}$

$$\int d^3\vec{x}\ \psi(\vec{x})\ e^{-ik\vec{x}} = \int d^3x \int \frac{d^3\vec{p}}{(2\pi)^3}\ e^{+i\vec{p}\vec{x}}\ a(p)\ e^{-ik\vec{x}}$$

then $\delta^{(3)}(k-p) = \int \frac{d^3\vec{x}}{(2\pi)^3}\ e^{-i(\vec{k}-\vec{p})\vec{x}}$

$$= \int d^3\vec{p}\ \delta(\vec{k}-\vec{p})\ a(p) = \int d^3\vec{p}\ \delta(\vec{p}-\vec{k})\ a(p) = a(k)$$

1

$$\psi^{+}(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3}\, e^{-i\vec{p}\cdot\vec{x}}\, a^{+}(\vec{p})$$

Let's integrate both sides on $\vec{x}$

$$\int d^3\vec{x}\;\psi^{+}(\vec{x}) = \int d^3x \int \frac{d^3\vec{p}}{(2\pi)^3}\; e^{-i\vec{p}\cdot\vec{x}}\; a^{+}(\vec{p})$$

$\cdot\, e^{i\vec{k}x}$

$$\int d^3\vec{x}\;\psi(\vec{x})\, e^{i\vec{k}\vec{x}} = \int d^3x \int \frac{d^3\vec{p}}{(2\pi)^3}\; e^{-i\vec{p}\vec{x}}\; a^{+}(p)\; e^{+ik\vec{x}}$$

$$= \int d^3\vec{p}\; \delta(p-k)\; a^{+}(p) = a^{+}(k)$$

Exercise 10 | QFT 9

Using the expression of $\varphi(\vec{x},t)$ show that

$$a(\vec{p}) = +i \int d^3\vec{x}\, e^{ipx} \overleftrightarrow{\partial_0} \varphi(\vec{x},t)$$

$$a^\dagger(\vec{p}) = -i \int d^3\vec{x}\, e^{-ipx} \overleftrightarrow{\partial_0} \varphi(\vec{x},t)$$

We know that

$$\varphi(\vec{x},t) = \int d\hat{p} \left[e^{-ip\cdot x} a(\vec{p}) + e^{ipx} a^\dagger(\vec{p}) \right]$$

We must recall that

- $f \overleftrightarrow{\partial} g = f \partial g - g \partial f$
- $\delta^{(3)}(\vec{p}) = \int_{-\infty}^{+\infty} e^{-i\vec{p}\vec{x}} d\vec{x}$
- $\mathcal{P} = (\omega_p, \vec{p})$, $\mathcal{K} = (\omega_k, \vec{k})$

Let's compute 3 useful relations

- $$\int d^3\vec{x}\, e^{-ipx} \frac{1}{(2\pi)^3} \frac{1}{2\omega_p} e^{ikx} = \int d^3\vec{x}\, \frac{\cancel{e^{-i\omega x_0}}}{(2\pi)^3} e^{i\vec{p}\vec{x}} \frac{1}{2\omega_p} \cancel{e^{i\omega x_0}} e^{-i\vec{k}\vec{x}} = \frac{1}{2\omega_p} \delta^{(3)}(-\vec{p}+\vec{k})$$
- $$\int d^3\vec{x}\, e^{ipx} \frac{1}{(2\pi)^3} \frac{1}{2\omega_p} e^{ikx} = \int d^3\vec{x}\, \frac{e^{i\omega_p x_0}}{(2\pi)^3} e^{-i\vec{p}\vec{x}} \frac{1}{2\omega_p} e^{i\omega_k x_0} e^{-i\vec{k}\vec{x}} = \frac{1}{2\omega_p} \delta^{(3)}(\vec{p}+\vec{k}) e^{2i(\omega_p+\omega_k)x_0}$$
- $$\int d^3\vec{x}\, e^{-ipx} \frac{1}{(2\pi)^3} \frac{1}{2\omega_p} e^{-ikx} = \int d^3\vec{x}\, \frac{e^{-i\omega_p x_0}}{(2\pi)^3} e^{+i\vec{p}\vec{x}} \frac{1}{2\omega_p} e^{-i\omega_k x_0} e^{+i\vec{k}\vec{x}} = \frac{1}{2\omega_p} \delta^{(3)}(-\vec{p}-\vec{k}) e^{-2i(\omega_p+\omega_k)x_0}$$

Now let's integrate the field by $\vec{x}$ and multiply by e^{-ikx}

$$\int d^3\vec{x}\, \varphi(x,t)\, e^{-ikx} = \int d^3\vec{x}\, e^{-ikx} \int \frac{d^3\vec{p}}{(2\pi)^3\, 2\omega_p}\, e^{-ipx}\, a(p) + \int d^3\vec{x}\, e^{-ikx} \int \frac{d^3\vec{p}}{(2\pi)^3\, 2\omega_p}\, e^{+ipx}\, a^\dagger(p)$$

$$= \int d^3\vec{p}\, \frac{1}{2\omega_p}\, \delta^{(3)}(-\vec{p}-\vec{k})\, e^{-2i(\omega_p+\omega_k)x_0}\, a(p) + \int d^3\vec{p}\, \frac{1}{2\omega_p}\, \delta^{(3)}(\vec{k}-\vec{p})\, a^\dagger(p)$$

$\omega_p = \omega_k$ because $\omega(\vec{k})$ depends on $\vec{k}$

$$= \frac{1}{2\omega_k}\, a(-\vec{k})\, e^{-2i(\omega_p+\omega_k)x_0} + \frac{1}{2\omega_k}\, a^\dagger(k)$$

Now let's compute

$$\int d^3\vec{x}\, e^{-ikx}\, \partial_0 \varphi(x) = \int d^3\vec{x}\, e^{-ikx} \int \frac{d^3\vec{p}}{(2\pi)^3\, 2\cancel{\omega_p}} \left[-i\cancel{\omega_p}\, e^{-ipx}\, a(p) + i\cancel{\omega_p}\, e^{+ipx}\, a^\dagger(p) \right]$$

= is the same as before without the "$\frac{1}{2\omega_p}$" in front and some "i" or "$-$"

$$= \frac{-i}{2}\, a(-\vec{k})\, e^{-2i(\omega_p+\omega_k)x_0} + \frac{i}{2}\, a^\dagger(k)$$

and so $\quad a^\dagger(\vec{k}) = -i \int d^3\vec{x}\, e^{-ikx}\, \overleftrightarrow{\partial_0}\, \varphi(x,t) = -i \int d^3\vec{x}\, e^{-ikx} \left[\partial_0 \varphi + i\omega_k \varphi \right] =$

$$= -i \left[\cancel{\frac{-i}{2}\, a(-\vec{k})\, e^{-2i(\omega_p+\omega_k)x_0}} + \frac{i}{2}\, a^\dagger(k) + i\, \cancel{\frac{\omega_k}{2\omega_k}\, a(-\vec{k})\, e^{-2i(\omega_p+\omega_k)x_0}} + i\, \frac{\cancel{\omega_k}}{2\cancel{\omega_k}}\, a^\dagger(k) \right]$$

$$= -i \cdot i\, a^\dagger(\vec{k}) = a^\dagger(\vec{k}) \quad \checkmark$$

For the $a(\vec{k})$ do the same but multiplying by e^{ikx} instead

EXERCISE 11 | QFT 9

$[a(\vec{p}), a(\vec{k})] = 0$

$$i\int d^3\vec{x}\, e^{ipx} \overleftrightarrow{\partial_0} \varphi(\vec{x},t) \cdot i\int d^3\vec{y}\, e^{iky} \overleftrightarrow{\partial_0} \varphi(\vec{y},t) - i\int d^3\vec{x}\, e^{ikx} \overleftrightarrow{\partial_0} \varphi(\vec{x},t) \cdot i\int d^3\vec{y}\, e^{ipy} \overleftrightarrow{\partial_0} \varphi(\vec{y},t) =$$

$$= i^2 \int d^3\vec{x}\, d^3\vec{y}\; e^{ipx} \overleftrightarrow{\partial_0} \varphi(\vec{x},t)\, e^{iky} \overleftrightarrow{\partial_0} \varphi(\vec{y},t) - i^2 \int d^3\vec{x}\, d^3\vec{y}\; \underbrace{e^{ikx} \overleftrightarrow{\partial_0} \varphi(\vec{x},t)}\, \underbrace{e^{ipy} \overleftrightarrow{\partial_0} \varphi(\vec{y},t)}$$

Let's compute the colored terms

$$e^{ikx} \overleftrightarrow{\partial_0} \varphi(\vec{x},t) = e^{ikx} \partial_0 \varphi - i\omega e^{ikx} \varphi \overset{!}{=} e^{ikx} \underbrace{[\partial_0 \varphi - i\omega \varphi]}_{1_x}$$

$$e^{ipy} \overleftrightarrow{\partial_0} \varphi(\vec{y},t) = e^{ipy} \partial_0 \varphi - i\omega e^{ipy} \varphi \overset{!}{=} e^{ipy} \underbrace{[\partial_0 \varphi - i\omega \varphi]}_{2_y}$$

After the multiplication we will have

$$e^{ikx} e^{ipy} [1_x] [2_y]$$

but if you exchange x and y

$$e^{iky} e^{ipx} [1_y] [2_x]$$

but it's the same integral so $[a, a^\dagger] = 0$

$[a^\dagger(k), a^\dagger(p)] = 0$

Same calculation as before but with + signs

$$\underline{[a(\vec{p}), a^\dagger(\vec{k})] = (2\pi)^3 (2\omega)\, \delta^{(3)}(\vec{p}-\vec{k})}$$

$$+i \int d^3\vec{x}\, e^{ipx} \overleftrightarrow{\partial_0} \varphi(\vec{x},t) \cdot -i \int d^3\vec{y}\, e^{-iky} \overleftrightarrow{\partial_0} \varphi(\vec{y},t) \; -$$

$$-i \int d^3\vec{x}\, e^{-ikx} \overleftrightarrow{\partial_0} \varphi(\vec{x},t) \cdot +i \int d^3\vec{y}\, e^{ipy} \overleftrightarrow{\partial_0} \varphi(\vec{y},t) \; =$$

$$= (-i)(+i) \int d^3\vec{x}\, d^3\vec{y}\;\; e^{ipx} \overleftrightarrow{\partial_0} \varphi(\vec{x},t)\, e^{-iky} \overleftrightarrow{\partial_0} \varphi(\vec{y},t) \; - \qquad (1)$$

$$- (+i)(-i) \int d^3\vec{x}\, d^3\vec{y}\;\; \underline{e^{-ikx} \overleftrightarrow{\partial_0} \varphi(\vec{x},t)}\, \underline{e^{ipy} \overleftrightarrow{\partial_0} \varphi(\vec{y},t)} \; = \qquad (2)$$

Change of variables to the 2° integral $x \leftrightarrow y$

$$= (-i)(+i) \int d^3\vec{x}\, d^3\vec{y}\;\; e^{ipx} \overleftrightarrow{\partial_0} \varphi(\vec{x},t)\, e^{-iky} \overleftrightarrow{\partial_0} \varphi(\vec{y},t) \; - \qquad (1)$$

$$- (+i)(-i) \int d^3\vec{y}\, d^3\vec{x}\;\; \underline{e^{-iky} \overleftrightarrow{\partial_0} \varphi(\vec{y},t)}\, \underline{e^{ipx} \overleftrightarrow{\partial_0} \varphi(\vec{x},t)} \; = \qquad (2)$$

$$= (+i)(-i) \int d^3\vec{x}\, d^3\vec{y}\;\; \left[e^{ipx} \overleftrightarrow{\partial_0} \varphi(\vec{x},t) \,,\, e^{-iky} \overleftrightarrow{\partial_0} \varphi(\vec{y},t) \right]$$

Let's rewrite $[e^{ipx}\overleftrightarrow{\partial_0}\varphi(\vec{x},t), e^{-iky}\overleftrightarrow{\partial_0}\varphi(\vec{y},t)]$

$$e^{iky}\overleftrightarrow{\partial_0}\varphi(\vec{y},t) = e^{iky}\partial_0\varphi - i\omega_k e^{iky}\varphi$$
$$= e^{iky}[\partial_0\varphi - i\omega_k\varphi] = e^{iky}(\pi(y) - i\omega_k\varphi(y))$$

$$e^{-ipx}\overleftrightarrow{\partial_0}\varphi(\vec{x},t) = e^{-ipx}\partial_0\varphi + i\omega e^{-ipx}\varphi$$
$$= e^{-ipx}[\partial_0\varphi + i\omega_p\varphi] = e^{-ipx}(\pi(x) + i\omega_p\varphi(x))$$

Now using

$$[a+b, c] = [a,c] + [b,c]$$

$$[a, b+c] = [a,b] + [a,c]$$

We get

$$[e^{iky}(\pi(y) - i\omega_k\varphi(y)), e^{-ipx}(\pi(x) + i\omega_p\varphi(x))] =$$

$$= [e^{iky}\pi(y), e^{-ipx}\pi(x)] + \longrightarrow \quad = e^{iky}e^{-ipx}\pi(y)\pi(x) - e^{-ikx}e^{iky}\pi(x)\pi(y)$$
$$= e^{iky}e^{-ipx}[\pi(y), \pi(x)] = 0$$

$$+ [e^{iky}\pi(y), e^{-ipx}(+i\omega_p\varphi(x))] \longrightarrow \quad = i\omega_p e^{ikx}e^{-ipy}[\pi(y), \varphi(x)]$$
$$= i\omega_p e^{ikx}e^{-ipy}[-i\delta^{(3)}(\vec{x}-\vec{y})]$$

$$+ [e^{iky}(-i\omega_k\varphi(y)), e^{-ipx}\pi(x)] \longrightarrow \quad = -i\omega_k e^{ikx}e^{-ipx}[i\delta^{(3)}(\vec{y}-\vec{x})]$$

$$+ [e^{iky}(-i\omega_k\varphi(y)), e^{-ipx}(+i\omega_p\varphi(x))] \qquad = 0$$

Hence

$$[a(\vec{p}), a^{+}(\vec{k})] = -i^{-2}\int d^3\vec{x} \int d^3 y \left\{ i w_p e^{ikx} e^{-ipy} \left[-i\, \delta^{(3)}(\vec{x}-\vec{y}) \right] \right\} +$$

$$-i^{-2}\int d^3\vec{x} \int d^3 y \left\{ -i w_k e^{ikx} e^{-ipx} \left[i\, \delta^{(3)}(\vec{y}-\vec{x}) \right] \right\} =$$

$$= -i^2 \int d^3\vec{y} \; i w_p e^{iky} e^{-ipy} \cdot -i \quad + \quad -i^2 \int d^3\vec{x} -i w_k e^{ikx} e^{-ipy} \cdot i$$

Since

$$(2\pi)^3 \delta(\vec{k}-\vec{p}) = \int d^3\vec{x}\, e^{-i(\vec{k}-\vec{p})x}$$

$$= -i^2 \int d^3\vec{y}\; e^{i w_k y_0} e^{-i\vec{k}\vec{y}} e^{-i w_p y_0} e^{+i\vec{p}\vec{y}} (-i)(i) w_p \quad + \quad -i^2 \int d^3\vec{x}\; e^{i w_k x_0} e^{-i\vec{k}\vec{x}} e^{-i w_p x_0} e^{+i\vec{p}\vec{x}} (+i)(-i) \cdot w_k$$

$$= -i^2 e^{i w_k y_0} e^{-i w_p y_0} w_p (2\pi)^3 \delta(\vec{k}-\vec{p}) \quad + \quad -i^2 e^{i w_k x_0} e^{-i w_p x_0} w_k (2\pi)^3 \delta(\vec{k}-\vec{p})$$

$$= \cancel{e^{i w_k y_0}} \cancel{e^{-i w_k y_0}} w_k (2\pi)^3 \delta(\vec{k}-\vec{p}) \quad + \quad \cancel{e^{i w_k x_0}} \cancel{e^{-i w_k x_0}} w_k (2\pi)^3 \delta(\vec{k}-\vec{p})$$

In the end

$$[a(\vec{p}), a^{+}(\vec{k})] = 2 w_k (2\pi)^3 \delta(\vec{k}-\vec{p})$$

but the delta is an even function: $\delta(x) = \delta(-x)$

So $\delta(\vec{p}-\vec{k}) = \delta(-\vec{p}+\vec{k}) = \delta(\vec{k}-\vec{p})$

$$\longrightarrow [a(\vec{p}), a^{+}(\vec{k})] = 2 w_k (2\pi)^3 \delta(\vec{p}-\vec{k}) \qquad \text{as required}$$

EXERCISE 12 | QFT 9

We know $\quad a(\vec{p}) = +i \int d^3\vec{x}\, e^{ipx} \overleftrightarrow{\partial_0}\, \varphi(\vec{x},t)$

$$a^{+}(\vec{p}) = -i \int d^3\vec{x}\, e^{-ipx} \overleftrightarrow{\partial_0}\, \varphi(\vec{x},t)$$

By defining $\quad N = \int d\hat{p}\; a^{+}(\vec{p})\, a(\vec{p}) \quad$ show that

1. $[N, a^{+}(\vec{k})] = a^{+}(\vec{k})$

$$= \int d\hat{p}\; a^{+}(p) a(p) a^{+}(k) - a^{+}(k) \int d\hat{p}\; a^{+}(p) a(p)$$

$$= \int d\hat{p}\; a^{+}(p) \underline{a(p) a^{+}(k)} - \int d\hat{p}\; a^{+}(k) a^{+}(p) a(p)$$

We know

- $[a(p), a^{+}(k)] = \underline{a(p) a^{+}(k)} - a^{+}(k) a(p) = (2\pi)^3 (2\omega_p)\, \delta(\vec{p}-\vec{k})$
- $[a^{+}(p), a^{+}(k)] = 0 = a^{+}(p) a^{+}(k) - a^{+}(k) a^{+}(p)$

$$= \int d\hat{p}\; a^{+}(p) \left(\underline{(2\pi)^3 2\omega_p\, \delta(\vec{p}-\vec{k}) + a^{+}(k) a(p)} \right) - \int d\hat{p}\; a^{+}(k) a^{+}(p) a(p)$$

$\rightarrow a^{+}(k) a^{+}(p) = a^{+}(p) a^{+}(k)$
$\rightarrow [a^{+}(k), a^{+}(p)] = 0$

$$= \int d\hat{p}\, (2\pi)^3 2\omega_p\, \delta(\vec{p}-\vec{k})\, a^{+}(p) + \cancel{\int d\hat{p}\; a^{+}(p) a^{+}(k) a(p)} - \cancel{\int d\hat{p}\; a^{+}(k) a^{+}(p) a(p)}$$

$$= \cancel{(2\pi)^3}\, \cancel{2\omega_p}\, \frac{1}{\cancel{(2\pi)^3}\, \cancel{2\omega_p}}\, a^{+}(\vec{k}) = a^{+}(k)$$

2. $[N, a(\vec{k})] = -a(\vec{k})$

$$= \int d\hat{p}\; a^{+}(p) a(p) a(k) - a(k) \int d\hat{p}\; a^{+}(p) a(p) = \int d\hat{p}\; \underbrace{a^{+}(p)\, \underline{a(p)\, a(k)}}_{a^{+}(p)\, \underline{a(k)\, a(p)}} - \int d\hat{p}\; a(k) a^{+}(p) a(p)$$

Because $[a(k), a(p)] = 0$

and then you develop $[a^{+}(p), a(k)]$

EXERCISE 13 | QFT 9

Substitute in H the expression of φ and π written in term of $a^\dagger(\vec{p})$ and $a(\vec{p})$ to show that

$$H = \int d\tilde{p} \; \frac{\omega_p}{2} \left[a^\dagger(\vec{p}) a(\vec{p}) + a(p) a^\dagger(\vec{p}) \right]$$

Starting from

$$H = \int d^3\vec{x} \, \frac{1}{2} \left[\pi^2 + (\vec{\nabla}\varphi)^2 + m^2 \varphi^2 \right]$$

Reminder

- $\varphi(x,t) = \int d\tilde{p} \left[e^{-ipx} a(\vec{p}) + e^{ipx} a^\dagger(\vec{p}) \right]$
- $\pi(\vec{x},t) = \partial_0 \varphi(\vec{x},t) = \int d\tilde{p} \left[-i\omega_p e^{-ipx} a(\vec{p}) + i\omega_p e^{ipx} a^\dagger(\vec{p}) \right]$
- $\int d\tilde{p} = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2\omega_p}$

Now let's divide the computation of H:

$$H = \int d^3\vec{x} \, \frac{1}{2} \left[\pi^2 + (\vec{\nabla}\varphi)^2 + m^2 \varphi^2 \right]$$

$$= \underbrace{\int d^3\vec{x} \, \frac{1}{2} \pi^2}_{1} + \underbrace{\int d^3x \, \frac{(\vec{\nabla}\varphi)^2}{2}}_{2} + \underbrace{\int d^3\vec{x} \, \frac{m^2}{2} \varphi^2}_{3} \Big]$$

$\boxed{1}$ $\int d^3x \, \frac{1}{2} [\pi^2]$

let's compute

$$\pi = \partial_0 \varphi = \int d\tilde{p} \, [-i\omega_p \, e^{-ipx} a(\vec{p}) + i\omega_p \, e^{ipx} a^\dagger(\vec{p})]$$

let's compute π^2

$$\pi \cdot \pi = \partial_0\varphi \cdot \partial_0\varphi = \int d\tilde{p} \int d\tilde{k} \, [-i\omega_p \, e^{-ipx} a(\vec{p}) \cdot -i\omega_k \, e^{-ikx} a(\vec{k}) +$$
$$-i\omega_p \, e^{-ipx} a(\vec{p}) \cdot +i\omega_k \, e^{ikx} a^\dagger(\vec{k}) +$$
$$i\omega_p \, e^{+ipx} a^\dagger(\vec{p}) \cdot -i\omega_k \, e^{-ikx} a(\vec{k}) +$$
$$i\omega_p \, e^{+ipx} a^\dagger(\vec{p}) \cdot +i\omega_k \, e^{ikx} a^\dagger(\vec{k})]$$

Now:

we know $\int d^3x \, e^{-i(p+k)x} = \delta^{(3)}(\vec{p}+\vec{k}) \, (2\pi)^3 \, e^{-i(\omega_p+\omega_k)}$

Follows

$$\int d^3x \, \pi^2 = \int d\tilde{p} \int d\tilde{k} \, [-i\omega_p \, -i\omega_k \, a(\vec{p}) a(\vec{k}) \, e^{-i(\omega_p+\omega_k)} (2\pi)^3 \, \delta(\vec{p}+\vec{k}) +$$
$$-i\omega_p \, +i\omega_k \, a(\vec{p}) a^\dagger(\vec{k}) \, e^{-i(\omega_p-\omega_k)} (2\pi)^3 \, \delta(\vec{p}-\vec{k}) +$$
$$+i\omega_p \, -i\omega_k \, a^\dagger(\vec{p}) a(\vec{k}) \, e^{+i(\omega_p-\omega_k)} (2\pi)^3 \, \delta(-\vec{p}+\vec{k}) +$$
$$+i\omega_p \, +i\omega_k \, a^\dagger(\vec{p}) a^\dagger(\vec{k}) \, e^{+i(\omega_p+\omega_k)} (2\pi)^3 \, \delta(-\vec{p}-\vec{k})] =$$

So $H = \frac{1}{2}\int d\hat{k}\ [-i\omega_k \cdot -i\omega_k\ a(-\vec{k})\, a(\vec{k})\, e^{-i(\omega_k+\omega_k)}\ (2\pi)^3\ +$

$-i\omega_k \cdot +i\omega_k\ a(\vec{k})\, a^\dagger(k) \cancel{e^{-i(\omega_k-\omega_k)}}\ (2\pi)^3\ +$

$+i\omega_k \cdot -i\omega_k\ a^\dagger(\vec{k})\, a(\vec{k}) \cancel{e^{-i(\omega_k-\omega_k)}}\ (2\pi)^3\ +$

$+i\omega_k \cdot +i\omega_k\ a^\dagger_{(-\vec{k})}\ a^\dagger(\vec{k})\, e^{+i(\omega_k+\omega_k)}\ (2\pi)^3\ ]$

Now i'll compute

2| $\int d^3x\ \frac{1}{2}(\vec{\nabla}\varphi)^2 =$

Let's compute first $\vec{\nabla}\varphi$

$$\vec{\nabla}\varphi = \int \frac{d\vec{p}}{(2\pi)^3}\frac{1}{2\omega_p}\left[\vec{\nabla}\left(e^{-ipx}a(\vec{p})\right) + \vec{\nabla}\left(e^{ipx}a^\dagger(\vec{p})\right)\right]$$

$$= \int d\hat{p}\ [\vec{\nabla}(e^{-ipx})\cdot a(\vec{p}) + \vec{\nabla}(e^{ipx})\, a^\dagger(p)]$$

$$= \int d\hat{p}\ [i p_i\, e^{-ipx}\, a(\vec{p}) + -ip_i\, e^{ipx}\, a^\dagger(\vec{p})]$$

Now the square

$$\vec{\nabla}\varphi\cdot\vec{\nabla}\varphi = \int d\hat{p}\int d\hat{k}\ [\ ip_i\, e^{-ipx}a(\vec{p})\cdot ik_i\, e^{-ikx}a(\vec{k})\ +$$

$$ip_i\, e^{-ipx}a(\vec{p})\cdot -ik_i\, e^{+ikx}a(\vec{k})\ +$$

$$-ip_i\, e^{ipx}a^\dagger(\vec{p})\cdot +ik_i\, e^{-ikx}a^\dagger(\vec{k})\ +$$

$$-ip_i\, e^{ipx}a^\dagger(\vec{p})\cdot -ik_i\, e^{ikx}a^\dagger(\vec{k})\ ]$$

Now using the $\int d\hat{x}$ integration and the delta

$$\int d^3x \, \frac{1}{2} (\vec{\nabla} \varphi)^2 =$$

$$\frac{1}{2} \int d\tilde{p} \int d\tilde{k} \, \Big[+i p_i \; +i k_i \; a(\vec{p}) \, a(\vec{k}) \, e^{+i(w_p + w_k)t} (2\pi)^3 \, \delta(-\vec{p}-\vec{k}) \; +$$

$$+i p_i \; - i k_i \; a(\vec{p}) \, a(\vec{k}) \, e^{+i(w_p - w_k)t} (2\pi)^3 \, \delta(-\vec{p}+\vec{k}) \; +$$

$$-i p_i \; +i k_i \; a^{+}(\vec{p}) \, a^{+}(\vec{k}) \, e^{-i(w_p - w_k)t} (2\pi)^3 \, \delta(\vec{p}-\vec{k}) \; +$$

$$-i p_i \; - i k_i \; a^{+}(\vec{p}) \, a^{+}(k) \, e^{-i(w_p + w_k)t} (2\pi)^3 \, \delta(\vec{p}+\vec{k}) \Big] =$$

$$= \int d\tilde{k} \, \frac{k^2}{2} \Big[a(\vec{k}) \, a(-\vec{k}) \, e^{-i 2 w_k t} (2\pi)^3 + a^{+}(k) \, a(k) \, (2\pi)^3 + a(k) \, a^{+}(k) \, (2\pi)^3 + a^{+}(k) \, a^{+}(-k) \, e^{i 2 w_k t} (2\pi)^3$$

3] $$\int d^3\vec{x} \, \frac{m^2}{2} \varphi^2 =$$

$$= \frac{m^2}{2} \int d^3\tilde{x} \left(\int d\tilde{p} \left[e^{-ipx} a(\vec{p}) + e^{ipx} a^{+}(\vec{p}) \right] \right) \left(\int d\tilde{k} \left[e^{-ikx} a(\vec{k}) + e^{ikx} a^{+}(\vec{k}) \right] \right)$$

$$= \frac{m^2}{2} \int d\tilde{k} \Big[\; a(-\vec{k}) \, a(\vec{k}) \, e^{-i 2 w_k} (2\pi)^3 \; +$$

$$a(\vec{k}) \, a^{+}(\vec{k}) \, (2\pi)^3 \; +$$

$$a^{+}(\vec{k}) \, a(k) \, (2\pi)^3 \; +$$

$$a^{+}(-\vec{k}) \, a^{+}(\vec{k}) \, e^{+i 2 w_k} (2\pi)^3 \; \Big]$$

So in the end

$$H = \int d\tilde{k} \Big\{ \cancel{\left(-\frac{\omega_k^2}{2} + \frac{k^2}{2} + \frac{m^2}{2} \right)} \cdot a\, a\, e^{-i \cdots} (2\pi)^3$$

$$+ \left(+\frac{\omega_k^2}{2} + \frac{k^2}{2} + \frac{m^2}{2} \right) \cdot a^{\dagger} a \quad (2\pi)^3$$

$$+ \left(+\frac{\omega_k^2}{2} + \frac{k^2}{2} + \frac{m^2}{2} \right) \cdot a\, a^{\dagger} \quad (2\pi)^3$$

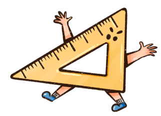
$$+ \cancel{\left(-\frac{\omega_k^2}{2} + \frac{k^2}{2} + \frac{m^2}{2} \right)} \cdot a^{\dagger} a^{\dagger}\, e^{+i \cdots} (2\pi)^3 \Big\}$$

$$\omega_k^2 = k^2 + m^2$$

In the end

$$H = \int \frac{d\vec{k}}{\cancel{(2\pi)^3}} \frac{\omega_k^{\cancel{2}}}{2\cancel{\omega_k}} \left[a^{\dagger}(\vec{k})\, a(\vec{k}) + a(k)\, a^{\dagger}(\vec{k}) \right] \cancel{(2\pi)^3}$$

$$= \int d^3\vec{k}\, \frac{\omega_k}{2} \left[a^{\dagger}(\vec{k})\, a(\vec{k}) + a(k)\, a^{\dagger}(\vec{k}) \right]$$



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$$j^\mu = :i\,\varphi^\dagger(\vec{x},t)\,\overleftrightarrow{\partial^\mu}\,\varphi(\vec{x},t):$$

and

$$Q = \int d^3x : i\,\varphi^\dagger \overleftrightarrow{\partial_0}\,\varphi :$$

Show

$$Q = \int d\tilde{p}\,\left[a^\dagger(\vec{p})\,a(\vec{p}) - b^\dagger(\vec{p})\,b(\vec{p})\right]$$

Let's recall that

$$\varphi(\vec{x},t) = \int d\hat{p}\left[e^{-ipx}a(p) + e^{ipx}b^\dagger(p)\right]$$

$$\varphi^\dagger(\vec{x},t) = \int d\hat{p}\left[e^{ipx}a^\dagger(p) + e^{-ipx}b(p)\right]$$

and

$$\varphi^\dagger\overleftrightarrow{\partial}\varphi = \varphi^\dagger\partial^0\varphi - \varphi\,\partial^0\varphi^\dagger$$

Let's compute

$$\varphi^\dagger\partial^0\varphi = \int d\hat{p}\int d\hat{k}\left[e^{ipx}a^\dagger(p) + e^{-ipx}b(p)\right]\left[a(k)(-i\omega_k)e^{-ikx} + b^\dagger(k)(i\omega_k)e^{ikx}\right]$$

$$= \int d\hat{p}\int d\hat{k}\left[e^{ipx}e^{-ikx}(-i\omega_k)\,a^\dagger(p)a(k) + e^{ipx}e^{ikx}(i\omega_k)\,a^\dagger(p)b^\dagger(k) + e^{-ipx}e^{-ikx}(-i\omega_k)\,b(p)a(k) + e^{-ipx}e^{ikx}(i\omega_k)\,b(p)b^\dagger(k)\right]$$

$$\varphi\,\partial^0\varphi^\dagger = \int d\hat{p}\int d\hat{k}\left[e^{-ipx}a(p) + e^{ipx}b^\dagger(p)\right]\left[a^\dagger(k)(i\omega_k)e^{ikx} + b(k)(-i\omega_k)e^{-ikx}\right]$$

$$= \int d\hat{p}\int d\hat{k}\left[e^{-ipx}e^{+ikx}(+i\omega_k)\,a(p)a^\dagger(k) + e^{-ipx}e^{-ikx}(-i\omega_k)\,a(p)b(k) + e^{+ipx}e^{+ikx}(+i\omega_k)\,b^\dagger(p)a^\dagger(k) + e^{+ipx}e^{-ikx}(-i\omega_k)\,b^\dagger(p)b(k)\right]$$

Now with integration over $d^3\vec{x}$

$$Q = \int d^3x \Bigg\{ \int d\hat{p} \int d\hat{k} \Big[e^{ipx} e^{-ikx} (-i\omega_k)\, a^\dagger(p)\, a(k) + e^{ipx} e^{ikx} \cancel{(i\omega_k)\, a^\dagger(p)\, b^\dagger(k)} + e^{-ipx} e^{-ikx} (-i\omega_k)\, \cancel{b(p)}\, a(k) + e^{-ipx} e^{ikx} (i\omega_k)\, b(p)\, b^\dagger(k) \Big] -$$

$$\Big(\int d\hat{p} \int d\hat{k} \Big[e^{-ipx} e^{+ikx} (+i\omega_k)\, a(p)\, a^\dagger(k) + e^{-ipx} e^{-ikx} (-i\omega_k)\, a(p)\, b(k) + e^{+ipx} e^{+ikx} (+i\omega_k)\, b^\dagger(p)\, a^\dagger(k) + e^{+ipx} e^{-ikx} (-i\omega_k)\, b^\dagger(p)\, b(k) \Big] \Big) \Bigg\}$$

$$\Big(\int d\hat{k} \int d\hat{p} \Big[e^{-ikx} e^{+ipx} (+i\omega_p)\, a(k)\, a^\dagger(p) + e^{-ikx} e^{-ipx} \cancel{(-i\omega_p)\, a(k)\, b(p)} + e^{+ikx} e^{+ipx} (+i\omega_p)\, \cancel{b^\dagger(k)}\, a^\dagger(p) + e^{+ikx} e^{-ipx} (-i\omega_p)\, b^\dagger(k)\, b(p) \Big] \Big) \Bigg\}$$

What this change of variable is now clear that, after using integration over d^3x the terms × cancels out cause $\omega_k = \omega_p$

Then

$$Q = \int d\hat{k} \frac{d^3\vec{p}}{(2\pi)^3 2\omega_p} \Big\{ \big[(-i\omega_k)\, a^\dagger(p)\, a(k) + (-i\omega_p)\, a(k)\, a^\dagger(p) \big] (2\pi)^3 \delta(\vec{p} - \vec{k}) + \big[(i\omega_k)\, b(p)\, b^\dagger(k) + (i\omega_p)\, b^\dagger(k)\, b(p) \big] (2\pi)^3 \delta(\vec{k} - \vec{p}) \Big\}$$

$$Q = \int d\hat{k} \frac{1}{\cancel{(2\pi)^3}\, 2\omega_k} \Big\{ \big[(-i\cancel{\omega_k})\, a^\dagger(k)\, a(k) + (-i\cancel{\omega_k})\, a(k)\, a^\dagger(k) \big] \cancel{(2\pi)^3} + \big[(i\cancel{\omega_k})\, b(k)\, b^\dagger(k) + (i\cancel{\omega_k})\, b^\dagger(k)\, b(k) \big] \cancel{(2\pi)^3} \Big\}$$

cancel out

2° pass

$$= \int d^3\hat{k} : i \big[-a^\dagger(k)\, a(k) + b^\dagger(k)\, b(k) \big] :$$

EXERCISE 13.1 | Pag 81

EXERCISE 13.2 | PAG 83

$$\mathcal{L} = \bar{\Psi}(i\partial_\mu \gamma^\mu - m)\Psi$$

From Nöther theorem there is a Nöther current that is conserved J^μ.

Since 2 Fields

$$J^\mu = \delta x^\mu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)}\delta\Psi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Psi})}\delta\bar{\Psi}$$

Infinitesimal case

$$e^{i\alpha} \simeq 1 + i\alpha \qquad e^{-i\alpha} \simeq 1 - i\alpha$$

Now

$$\delta\Psi(x) = \Psi'(x) - \Psi(x) = i\alpha\Psi(x)$$

$$\delta\bar{\Psi}(x) = \bar{\Psi}'(x) - \Psi(x) = -i\alpha\bar{\Psi}(x)$$

And $\delta x = 0$ since $U(1)$ transformation doesn't act on space time

---

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} = \bar{\Psi} i \gamma^\mu \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Psi})} = 0$$

So $$J^\mu = -\bar{\Psi} i \gamma^\mu (i\alpha\Psi(x)) = +\alpha\bar{\Psi}\gamma^\mu\Psi \Rightarrow j^\mu = :\bar{\Psi}\gamma^\mu\Psi:$$

2° prescription

Exercise 13.31 QFT 11 pag 83

2) Show Q

$$\Psi(\vec{x},t) = \sum_{s=1,2} \int d\tilde{p} \left[b_s(\vec{p}) u_\alpha^s(p) e^{-ipx} + d_s^\dagger(\vec{p}) v_\alpha^s(p) e^{ipx} \right]$$

$$\bar{\Psi}(\vec{x},t) = \sum_{s'=1,2} \int d\tilde{p}' \left[b_{s'}^\dagger(\vec{p}') \bar{u}_\alpha^{s'}(p') e^{+ip'x} + d_{s'}(\vec{p}) \bar{v}_\alpha^{s'}(\vec{p}') e^{-ip'x} \right]$$

Now $Q = \int d^3x : \bar{\Psi} \gamma^0 \Psi :$

$$Q = \int d^3x : \Big[\int d\tilde{p} \left[b_s(\vec{p}) u_\alpha^s(p) e^{-ipx} + d_s^\dagger(\vec{p}) v_\alpha^s(p) e^{ipx} \right] \gamma^0 \cdot$$
$$\int d\tilde{p}' \left[b_{s'}^\dagger(\vec{p}') \bar{u}_\alpha^{s'}(p') e^{+ip'x} + d_{s'}(\vec{p}) \bar{v}_\alpha^{s'}(\vec{p}') e^{-ip'x} \right] \Big] :$$

$$= \int d\tilde{p} \int d\tilde{p}' \int d^3x : \Big[b_s(\vec{p}) u_\alpha^s(p) e^{-ipx} \gamma^0 b_{s'}^\dagger(\vec{p}') \bar{u}_\alpha^{s'}(p') e^{+ip'x}$$
$$b_s(\vec{p}) u_\alpha^s(p) e^{-ipx} \gamma^0 d_{s'}(\vec{p}) \bar{v}_\alpha^{s'}(\vec{p}') e^{-ip'x}$$
$$d_s^\dagger(\vec{p}) v_\alpha^s(p) e^{ipx} \gamma^0 b_{s'}^\dagger(\vec{p}') \bar{u}_\alpha^{s'}(p') e^{+ip'x}$$
$$d_s^\dagger(\vec{p}) v_\alpha^s(p) e^{ipx} \gamma^0 d_{s'}(\vec{p}) \bar{v}_\alpha^{s'}(\vec{p}') e^{-ip'x} \Big] :$$

Integration over x

$$\int d^3x\, e^{-ipx} e^{+ip'x} = \int d^3x\, e^{-ip_0x_0} e^{+ip_0'x_0} e^{i\vec{p}\vec{x}} e^{-i\vec{p}'\vec{x}} = e^{-i(p_0-p_0')} (2\pi)^3 \delta^{(3)}(\vec{p}'-\vec{p})$$

$$- \quad - \quad = \qquad = e^{-i(-p_0-p_0')} (2\pi)^3 \delta^{(3)}(-\vec{p}'-\vec{p})$$

$$+ \quad + \quad = \qquad = e^{-i(p_0+p_0')} (2\pi)^3 \delta^{(3)}(\vec{p}'+\vec{p})$$

$$+ \quad - \quad = \qquad = e^{-i(-p_0+p_0')} (2\pi)^3 \delta(-\vec{p}'+\vec{p})$$

$$= \int d\tilde{p} \int d\tilde{p}' :\Big[b_s(\vec{p}) u_a^s(p)\, \gamma^0\, b_{s'}^{+}(\vec{p}') \bar{u}_a^{s'}(p')\, e^{-i(p_0 - p_0')} (2\pi)^3 \delta^{(3)}(\vec{p}' - \vec{p})$$

$$b_s(\vec{p}) u_a^s(p)\, \gamma^0\, d_{s'}(\vec{p}') \bar{v}_a^{s'}(\vec{p}')\, e^{-i(-p_0 - p_0')} (2\pi)^3 \delta^{(3)}(-\vec{p}' - \vec{p})$$

$$d_s^{+}(\vec{p}) v_a^s(p)\, \gamma^0\, b_{s'}^{+}(\vec{p}') \bar{u}_a^{s'}(p')\, e^{-i(p_0 + p_0')} (2\pi)^3 \delta^{(3)}(\vec{p}' + \vec{p})$$

$$d_s^{+}(\vec{p}) v_a^s(p)\, \gamma^0\, d_{s'}(\vec{p}') \bar{v}_a^{s'}(\vec{p}')\, e^{-i(-p_0 + p_0')} (2\pi)^3 \delta(-\vec{p}' + \vec{p}) \Big]:$$

Also the expotentials go away $\Rightarrow$ $p_0'^2 = m^2 + \vec{p}'^2$ $\hookrightarrow$ This is $\vec{p}^2$

$$= \int d\tilde{p} \frac{1}{2w_{p'}} \frac{1}{(2\pi)^3} (2\pi)^2 :\Big[b_s(\vec{p}) u_a^s(p)\, \gamma^0\, b_{s'}^{+}(\vec{p}) \bar{u}_a^{s'}(p)\, \cancel{e^{-i(p_0 - p_0')}}$$

$$\left. \begin{aligned} &b_s(\vec{p}) u_a^s(p)\, \gamma^0\, d_{s'}(-\vec{p}) \bar{v}_a^{s'}(-\vec{p})\, e^{-i(-p_0 - p_0')} \end{aligned} \right\} = 0$$

$$\left. \begin{aligned} &d_s^{+}(\vec{p}) v_a^s(p)\, \gamma^0\, b_{s'}^{+}(-\vec{p}) \bar{u}_a^{s'}(-\vec{p})\, e^{-i(p_0 + p_0')} \end{aligned} \right\} = 0$$

$$d_s^{+}(\vec{p}) v_a^s(p)\, \gamma^0\, d_{s'}(\vec{p}) \bar{v}_a^{s'}(\vec{p})\, \cancel{e^{-i(-p_0 + p_0')}} \Big]:$$

$$= \int d\tilde{p} \frac{1}{2w_p} \frac{1}{(2\pi)} :\Big[b_s(\vec{p}) b_{s'}^{+}(\vec{p})\, \cancel{2w_p}\, \delta_{ss'} + d_s^{+}(\vec{p}) d_{s'}(\vec{p})\, \cancel{2w_p}\, \delta_{ss'} \Big]:$$

$$\overset{!}{=} \int d\tilde{p} \frac{1}{2\pi} \left[-b_s^{+}(\vec{p}) b_s(\vec{p}) + d_s^{+}(\vec{p}) d_s(\vec{p}) \right]$$ sumed on the spin s

Exercise 14 | QFT 14

Double Fourier Transformation

$$\int d^4x\, d^4y \; G_F(x,y)^2 \, e^{i(P_1+P_2)x - i(K_1+K_2)y} \xrightarrow{\text{Double Fourier}} (2\pi)^4 \delta^{(4)}(K_1+K_2-P_1-P_2) \int \frac{d^4K}{(2\pi)^4} \tilde{G}_F(K)\, \tilde{G}_F(K_1+K_2-K)$$

We know that $G_F(x,y) = G_F(x-y)$, so

$$G(x-y) = \int \frac{d^4\vec{K}}{(2\pi)^4} \tilde{G}(K)\, e^{-i(\vec{x}-\vec{y})\vec{K}}$$

So

$$= \int d^4x\, d^4y \; G_F(x,y)\, G_F(x,y)\, e^{i(P_1+P_2)x - i(K_1+K_2)y}$$

$$= \int d^4x\, d^4y \; G_F(x,y)\, e^{i(P_1+P_2)x - i(K_1+K_2)y} \int \frac{d^{3}\vec{K}}{(2\pi)^3} \tilde{G}(K)\, e^{-i(\vec{x}-\vec{y})\vec{K}}$$

Now, since

$$\tilde{G}(m,\ell) = \int d^4x\, d^4y \; G(x,y)\, e^{ixm}\, e^{iy\ell}$$

$$m = P_1 + P_2 - K$$
$$\ell = K - K_1 - K_2$$

And so

$$= \int \frac{d^{3}\vec{K}}{(2\pi)^3} \tilde{G}(P_1+P_2-K,\, K-K_1-K_2)\, \tilde{G}(K)$$

Now imposing the 4-mom conservation $\delta^{(4)}(K_1+K_2-P_1-P_2)$

We will have

$$\tilde{G}(P_1+P_2-K,\, K-K_1-K_2) = \tilde{G}(P_1+P_2-K,\, K-K_1-K_2)$$
$$= \tilde{G}(2K_1+2K_2-2K)$$

IMPORTANT EXERCISE 15 | QFT

Feynmann Trick for spin 1 photon Lagrangian: PROPAGATOR

$$\mathcal{L}_{KIN} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)(\partial_\mu A^\mu)$$

We need to compute $i \left[\frac{\partial^2 \mathcal{L}_{KIN}}{\partial A_\alpha \partial A_\beta} \Big|_{\substack{A_\alpha = 0 = A_\beta \\ p' = -p}} \right]^{-1}$

$$\frac{\partial^2 \mathcal{L}_{KIN}}{\partial A_\beta \partial A_\alpha} = -\frac{1}{4} \underbrace{\frac{\partial^2}{\partial A_\beta \partial A_\alpha} (F^{\mu\nu} F_{\mu\nu})}_{1° \text{ TERM}} - \frac{1}{2\xi} \underbrace{\frac{\partial^2}{\partial A_\beta \partial A_\alpha} \big((\partial_\mu A^\mu)(\partial_\mu A^\mu)\big)}_{2° \text{ TERM}}$$

1° TERM

Let's develop $F^{\mu\nu} F_{\mu\nu}$ Firstly

- $$F^{\mu\nu} F_{\mu\nu} = (\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$
$$= \underbrace{\partial^\mu A^\nu \partial_\mu A_\nu} - \underbrace{\partial^\mu A^\nu \partial_\nu A_\mu} - \underbrace{\partial^\nu A^\mu \partial_\mu A_\nu}_{\nu \to \mu} + \underbrace{\partial^\nu A^\mu \partial_\nu A_\mu}_{\nu \to \mu}$$
$$= 2(\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A_\mu)$$

First derivative

$$\frac{\partial F^{\mu\nu} F_{\mu\nu}}{\partial A_\alpha} = 2 \left(\frac{\partial (\partial^\mu A^\nu)}{\partial A_\alpha} \cdot \partial_\mu A_\nu + \partial^\mu A^\nu \cdot \frac{\partial (\partial_\mu A_\nu)}{\partial A_\alpha} - \frac{\partial (\partial^\mu A^\nu)}{\partial A_\alpha} \cdot \partial_\nu A_\mu - \partial^\mu A^\nu \frac{\partial (\partial_\nu A_\mu)}{\partial A_\alpha} \right)$$

We know $g^{\mu\nu} g_{\mu\nu} = \delta^\mu_\mu = 4$

Now we know the Trick

$$\frac{\partial}{\partial A_\alpha}(\partial_\nu A_\beta) \equiv -i p_\nu g^\alpha_\beta$$

$\longrightarrow \frac{\partial}{\partial A_\alpha}(\partial_\nu A_\mu) \equiv -i p_\nu g^\alpha_\mu$ ||| $\longrightarrow \frac{\partial}{\partial A_\alpha}(\partial^\mu A^\nu) \equiv -i p^\mu g^{\alpha\nu}$

$\longrightarrow \frac{\partial}{\partial A_\alpha}(\partial_\mu A_\nu) \equiv -i p_\mu g^\alpha_\nu$

This is True. If you write the upper index as lower times the metrix, then are derivatives goes away =)

$$\text{So} = 2\Big((-i p^\mu g^{\alpha\nu})\cdot\partial_\mu A_\nu + \partial^\mu A^\nu\cdot(-i p_\mu g^\alpha_\nu)$$

$$- (-i p^\mu g^{\alpha\nu})\cdot\partial_\nu A_\mu - \partial^\mu A^\nu(-i p_\nu g^\alpha_\mu)\Big) \qquad (*)$$

Second derivative

$$\frac{\partial *}{\partial A_\beta} = 2\Big((-i p^\mu g^{\alpha\nu})\cdot\frac{\partial(\partial_\mu A_\nu)}{\partial A_\beta} + \frac{\partial(\partial^\mu A^\nu)}{\partial A_\beta}\cdot(-i p_\mu g^\alpha_\nu)$$

$$- (-i p^\mu g^{\alpha\nu})\cdot\frac{\partial(\partial_\nu A_\mu)}{\partial A_\beta} - \frac{\partial(\partial^\mu A^\nu)}{\partial A_\beta}(-i p_\nu g^\alpha_\mu)\Big)$$

$\longrightarrow \frac{\partial}{\partial A_\beta}(\partial_\mu A_\nu) \equiv -i p'_\mu g^\beta_\nu$ ||| $\longrightarrow \frac{\partial}{\partial A_\beta}(\partial^\mu A^\nu) \equiv -i p'^\mu g^{\beta\nu}$

$\longrightarrow \frac{\partial}{\partial A_\beta}(\partial_\nu A_\mu) \equiv -i p'_\nu g^\beta_\mu$

$$\equiv 2\Big((-i p^\mu g^{\alpha\nu})\cdot(-i p'_\mu g^\beta_\nu) + (-i p'^\mu g^{\beta\nu})(-i p_\mu g^\alpha_\nu)$$

$$- (-i p^\mu g^{\alpha\nu})\cdot(-i p'_\nu\cdot g^\beta_\mu) - (-i p'^\mu g^{\beta\nu})(-i p_\nu g^\alpha_\mu)\Big)$$

$$= 2\Big[\big(-p^{\mu} g^{\alpha\nu} p'_{\mu} g^{\beta}_{\ \nu}\big) + \big(-p'^{\mu} g^{\beta\nu} p_{\mu} g^{\alpha}_{\ \nu}\big)$$

$$-\big(-p^{\mu} g^{\alpha\nu} p'_{\nu} g^{\beta}_{\ \mu}\big) + \big(-p'^{\mu} g^{\beta\nu} p_{\nu} g^{\alpha}_{\ \mu}\big)$$

$$= 2\Big[-p^{\mu} p'_{\mu} g^{\alpha\beta} - p'^{\mu} p_{\mu} g^{\beta\alpha}$$

$$-p^{\beta} p'^{\alpha} - p'^{\alpha} p^{\alpha}\Big]$$

$$= 2\big[-2g^{\alpha\beta}\, p p' - p^{\beta} p'^{\alpha} - p'^{\alpha} p^{\beta}\big]$$

2° TERM

$$\frac{\partial^2}{\partial A_\alpha \partial A_\beta}\Big((\partial_\mu A^\mu)(\partial_\mu A^\mu)\Big)$$

First Derivative

$$\frac{\partial}{\partial A_\beta}\Big((\partial_\mu A^\mu)(\partial_\mu A^\mu)\Big) = \frac{\partial}{\partial A_\beta}(\partial_\mu A^\mu)\cdot\partial_\mu A^\mu + \partial_\mu A^\mu \frac{\partial}{\partial A_\beta}(\partial_\mu A^\mu)$$

$$\equiv (-i p_\mu g^{\beta\mu})(\partial_\mu A^\mu) + \partial_\mu A^\mu(-i p_\mu g^{\beta\mu}) \qquad (*)$$

Second Derivative

$$\frac{\partial}{\partial A_\alpha}(*) = (-i p_\mu g^{\beta\mu})\Big(\frac{\partial}{\partial A_\alpha}(\partial_\mu A^\mu)\Big) + \Big(\frac{\partial}{\partial A_\alpha}(\partial_\mu A^\mu)\Big)(-i p_\mu g^{\beta\mu})$$

$$\equiv (-i p_\mu g^{\beta\mu})(-i p'_\mu g^{\alpha\mu}) + (-i p'_\mu g^{\alpha\mu})(-i p_\mu g^{\beta\mu})$$

$$= -p^{\beta} p'^{\alpha} - p'^{\alpha} p^{\beta}$$

All toghter

$$\frac{\partial^2 \mathcal{L}_{KIN}}{\partial A_\mu \partial A_\nu} = -\frac{1}{4} \underbrace{\frac{\partial^2}{\partial A_\beta \partial A_\alpha} (F^{\mu\nu} F_{\mu\nu})}_{1^\circ \text{ TERM}} - \frac{1}{2\xi} \underbrace{\frac{\partial^2}{\partial A_\beta \partial A_\alpha} \left((\partial_\mu A^\mu)(\partial_\mu A^\mu) \right)}_{2^\circ \text{ TERM}}$$

$$= -\frac{1}{4} \left[2[-2g^{\alpha\beta} p p' - p^\beta p'^\alpha - p'^\alpha p^\beta] \right] - \frac{1}{2\xi} \left[-p^\alpha p'^\beta - p'^\beta p^\alpha \right]$$

Field $= 0$

$p' = -p$

$$= -\frac{1}{4} \left[2 [+2g^{\alpha\beta} p^2 \underbrace{+ p^\beta p^\alpha + p^\alpha p^\beta}] \right] - \frac{1}{\cancel{2}\xi} \cdot \cancel{2}\, p^\alpha p^\beta$$

$p^\alpha p_\beta = p^\beta p_\alpha \rightarrow p_\beta g^{\beta\alpha} p_\beta = p_\alpha g^{\alpha\beta} p_\alpha$

$$= -\frac{1}{\cancel{4}} \cdot \cancel{4} \left[g^{\alpha\beta} p^2 + p^\alpha p^\beta \right] - \frac{1}{\xi} p^\alpha p^\beta$$

$$= -g^{\alpha\beta} p^2 + \left(1 - \frac{1}{\xi}\right) p^\alpha p^\beta$$

Now invert and multiply by i

$$i \left[-g^{\alpha\beta} p^2 + \left(1 - \frac{1}{\xi}\right) p^\alpha p^\beta \right]^{-1}$$

We now start from $\left(-p^2 g_{\alpha\beta} + \left(1 - \frac{1}{\xi}\right) p_\alpha p_\beta \right) D^{\nu\rho}(p) = \delta^\rho_\mu$ (1)

And we impose $D^{\mu\rho}(p) = A g^{\mu\nu} + B p^\mu p^\rho$ (2)

Inserting (2) in (1) we get

$$\left[-p^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right) p_\mu p_\nu \right] \left[A g^{\nu\rho} + B p^\nu p^\rho \right] = i \delta^\rho_\mu$$

Now by expanding and comparing the terms by isolation we get

$$A = -\frac{1}{p^2} \qquad B = \frac{i(1-\xi)}{p^4}$$

Exercise 16 | QFT 18

Show that for $A+B \to C+D$

$$d\varphi^{(2)} = \frac{1}{(2\pi)^2} \frac{\|P\|}{4\sqrt{s}} d\Omega$$

$$d\varphi^{(2)} = \frac{1}{(2\pi)^2} \frac{1}{\omega_3 \omega_4} \delta(\omega - \omega_3 - \omega_4)\, \delta^3(P_3 + P_4)\, d^3P_3\, d^3P_4$$

Now we integrate over d^3P_4

$$d\varphi^{(2)}_{NEW} = \int_0^{+\infty} \frac{1}{(2\pi)^2} \frac{1}{\omega_3 \omega_4} \delta(\omega - \omega_3 - \omega_4)\, \delta^3(P_3 + P_4)\, d^3P_3\, d^3P_4$$

$$= \frac{1}{(2\pi)^2} \frac{1}{\omega_3 \omega_4} \delta(\omega - \omega_3 - \omega_4)\, d^3P_3$$

but with $\omega_4 = \sqrt{m_4^2 + P_4^2}$ → $\omega_4 = \sqrt{m_4^2 + P_3^2}$

Now the integration over d^3P_3, using spherical coordinates

$$d^3P_3 = P_3^2\, dP_3\, d\Omega$$

So

$$d\varphi_{NEW} = \int_0^{+\infty} \frac{1}{(2\pi)^2} \frac{1}{\omega_3 \omega_4} \delta(\omega - \omega_3 - \omega_4)\, P_3^2\, dP_3\, d\Omega$$

$$= \int_0^{+\infty} \frac{1}{(2\pi)^2} \frac{1}{\sqrt{m_3^2 + P_3^2}} \frac{1}{\sqrt{m_4^2 + P_3^2}} \frac{\delta(\omega - \omega_3 - \omega_4)\, P_3^2\, dP_3\, d\Omega}{2 \cdot 2}$$

Change of variable to solve the integral

$$\omega_i^2 = m_i^2 + P_i^2 \quad \leadsto \quad 2\omega_i\, d\omega_i = 2P_i\, dP_i$$

So $\quad dW_3 + dW_4 = \frac{2P_3\,dP_3}{2W_3} + \frac{2P_3\,dP_3}{2W_4} = dW'$

$= P_3 dP_3 \left(\frac{1}{W_3} + \frac{1}{W_4}\right)$

$\Rightarrow P_3 dP_3 = \frac{W_3 W_4}{W'} dW' \quad \Rightarrow$

So

$$= \int_0^{+\infty} \frac{1}{4(2\pi)^2} \cancel{\frac{1}{\sqrt{m_3^2+P_3^2}}} \cancel{\frac{1}{\sqrt{m_4^2+P_3^2}}} \delta(W-W')\, P_3 \frac{\cancel{W_3 W_4}}{W'} dW'\, d\Omega$$

$$= \frac{1}{(2\pi)^2} \frac{1}{4} \frac{P_3(W)}{W} d\Omega$$

EXERCISE 27 | QFT 18

Show that for $A+B \to C+D \qquad \underline{\Phi} = 4\,\|P\|\sqrt{S}$

Definition:

$$\underline{\Phi}^{CM} = 4\cdot\sqrt{(P_2 \cdot P_2)^2 - m_2^2 m_2^2}$$

If we develop $(P_2 \cdot P_2)^2 = \left((W_2 W_2) - (\vec{P}_2 \cdot \vec{P}_2)\right)^2 \qquad$ in the CM $\quad P_2 = P_2 = P$

$= (W_2 W_2 - \vec{P}^2)^2$

$= (W_2^2 W_2^2 + \vec{P}^4 + 2\vec{P}^2 W_2 W_2)$

$= (\vec{P}^2 + m_2^2)(\vec{P}^2 + m_2^2) + 2\vec{P}^2 W_2 W_2 + \vec{P}^4 = \vec{P}^2 (2P_2^2 + m_2^2 + m_2^2 + 2W_2 W_2) + m_1^2 m_2^2$

Since $\quad S = (W_2 + W_2)^2 = W_2^2 + W_2^2 + 2W_2W_2 = P_2^2 + m_2^2 + P_2^2 + m_2^2 + 2W_2W_2 = 2\vec{P}^2 + m_2^2 + m_2^2 + 2W_2W_2$

So $\quad \underline{\Phi}^{CM} = 4\cdot\sqrt{\vec{P}^2(S) + m_1^2 m_2^2 - m_1^2 m_2^2} = 4\,\|\vec{P}\|\,S$

EXERCISE

$$Q = \int d^3\vec{x} :\left(\bar{\psi}(\vec{x},t)\,\gamma^0\,\psi(\vec{x},t)\right): = \sum_{t,s}\int d\tilde{p}\,\frac{1}{2\omega_p}\Big[\, b_t^{\dagger}(\vec{p})\,\overbrace{\bar{u}_a^{t}(\vec{p})\,\gamma^0}\, b_s(\vec{p})\,\overbrace{u_a^{s}(\vec{p})}^{\text{normalization}} + b_t^{\dagger}(-\vec{p})\,\overbrace{\bar{u}_a^{t}(-\vec{p})\,\gamma^0}\, d_s^{\dagger}(\vec{p})\,\overbrace{v_a^{s}(\vec{p})}^{0}\, e^{i2\omega_p t}$$

$$+\, d_t(-\vec{p})\,\bar{v}_a^{t}(-\vec{p})\,\gamma^0\, b_s(\vec{p})\,u_a^{s}(\vec{p})\; e^{-i2\omega_p t} \; +$$

$$+\, d_t(\vec{p})\,\bar{v}_a^{t}(\vec{p})\,\gamma^0\, d_s^{\dagger}(\vec{p})\,v_a^{s}(\vec{p})\Big]: \quad =$$

$$= \sum_{t,s}\int d\tilde{p}\,\frac{1}{\cancel{2\omega_p}} :\Big[\, b_t^{\dagger}(\vec{p})\, b_s(\vec{p})\,\cancel{2\omega_p}\,\delta^{st} + \underbrace{d_t(\vec{p})\, d_s^{\dagger}(\vec{p})}_{-\text{ sign for } ::}\,\cancel{2\omega_p}\,\delta^{ts}\Big]:$$

Exercise

$$H = \bar{\Psi}(i\vec{\gamma}\cdot\vec{\nabla} + m)\Psi$$

$$\Psi_a(\vec{x},t) = \sum_{s=1,2}\int d\tilde{p}\,\left[b_s(\vec{p})\,u_a^s(\vec{p})\,e^{-ipx} + d_s^+(\vec{p})\,v_a^s(\vec{p})\,e^{ipx}\right]$$

$$\bar{\Psi}_a(\vec{x},t) = \sum_{t=1,2}\int d\tilde{q}\,\left[b_t^+(\vec{q})\,\bar{u}_a^t(\vec{q})\,e^{iqx} + d_t(\vec{q})\,\bar{v}_a^t(\vec{q})\,e^{-iqx}\right]$$

$$\vec{\nabla}\Psi_a(\vec{x},t) = \sum_{s=1,2}\int d\tilde{p}\,\left[b_s(\vec{p})\,u_a^s(\vec{p})\,(i\vec{p})\,e^{-ipx} + d_s^+(\vec{p})\,v_a^s(\vec{p})\,(-i\vec{p})\,e^{ipx}\right]$$

$$\vec{\gamma}\cdot\vec{\nabla}\Psi_a(\vec{x},t) = \sum_{s=1,2}\int d\hat{p}\;i\vec{\gamma}\cdot\vec{p}\,\left[b_s(\vec{p})\,u_a^s(\vec{p})\,e^{-ipx} - d_s^+(\vec{p})\,v_a^s(\vec{p})\,e^{ipx}\right]$$

$$i\int d^3x\;\bar{\Psi}\vec{\gamma}\cdot\vec{\nabla}\Psi = \int d^3x\sum_{t=1,2}\int d\hat{q}\,\left(b_t^+(\vec{q})\,\bar{u}_a^t(\vec{q})\,e^{iqt} + d_t(\vec{q})\,\bar{v}_a^t(\vec{q})\,e^{-iqx}\right]\cdot$$

$$\sum_{s=1,2}\int d\hat{p}\;i\;\vec{\gamma}\cdot\vec{p}\,\left[b_s(\vec{p})\,u_a^s(\vec{p})\,e^{-ipx} - d_s^+(\vec{p})\,v_a^s(\vec{p})\,e^{ipx}\right]i$$

$$= \sum_t\sum_s\int\frac{d^3\vec{q}}{(2\pi)^3\,2w_q}\int d\tilde{p}\;\;i\;\vec{\gamma}\cdot\vec{p}\,\Big[b_t^+(\vec{q})\,\bar{u}_a^t(\vec{q})\;b_s(\vec{p})\,u_a^s(\vec{p})\,e^{i(q-p)x} - b_t^+(\vec{q})\,\bar{u}_a^t(\vec{q})\,d_s^+(\vec{p})\,v_a^s(\vec{p})\,e^{i(q+p)x} +$$

$$+ d_t(\vec{q})\,\bar{v}_a^t(\vec{q})\,b_s(\vec{p})\,u_a^s(\vec{p})\,e^{-i(q+p)x} - d_t(\vec{q})\,\bar{v}_a^t(\vec{q})\,d_s^+(\vec{p})\,v_a^s(\vec{p})\,e^{-i(q-p)x}\Big]i$$

$$= -\sum_{t,s}\frac{1}{2w_p}\int d\hat{p}\;\vec{\gamma}\cdot\vec{p}\,\Big[b_t^+(\vec{p})\,\bar{u}_a^t\,b_s(\vec{p})\,u_a^s(\vec{p}) - b_t^+(-\vec{p})\,\bar{u}_a^t(-\vec{p})\;d_s^+(\vec{p})\,v_a^s(\vec{p})\,e^{i2w_pt} +$$

$$d_t(-\vec{p})\;\bar{v}_a^t(-\vec{p})\;b_s(\vec{p})\,u_a^s(\vec{p})\;e^{-2iw_pt} - d_t(\vec{p})\,\bar{v}_a^t(\vec{p})\,d_s^+(\vec{p})\;v_a^s(\vec{p})\Big]$$

$$m\int d^3x\,\bar{\Psi}\Psi = \sum_{t,s}\int d\hat{p}\,\frac{m}{2w_p}\;\Big[b_t^+(\vec{p})\,\bar{u}_a^t\,b_s(\vec{p})\,u_a^s(\vec{p}) + b_t^+(-\vec{p})\,\bar{v}_a^t(-\vec{p})\;d_s^+(\vec{p})\,v_a^s(\vec{p})\;e^{i2w_pt} +$$

$$+ d_t(-\vec{p})\;\bar{v}_a^t(-\vec{p})\;b_s(\vec{p})\,u_a^s(\vec{p})\;e^{-2iw_pt} + d_t(\vec{p})\,\bar{v}_a^t(\vec{p})\,d_s^+(\vec{p})\,v_a^s(\vec{p})\Big]$$

Then

$$\bar{u}_\lambda^t(-p)\,[\vec{\gamma}\cdot\vec{p}+m]\,v_\lambda^s(\vec{p}) = \bar{u}_\lambda^t(-\vec{p})\,(-\gamma^0 p_0)\,v_\lambda^s(\vec{p}) = 0$$

$$\bar{v}_\lambda^t(-p)\,[-\vec{\gamma}\cdot\vec{p}+m]\,u_\lambda^s(\vec{p}) = \bar{v}_\lambda^t(-\vec{p})\,\gamma^0 p_0\, u_\lambda^s(\vec{p}) = 0$$

Hence

$$H = \sum_{t,s}\frac{1}{2\omega_p}\int d\hat{p}\,\Big\{\big[b_t^\dagger(\vec{p})\underbrace{\bar{u}_\lambda^t(\vec{p})[-\vec{\gamma}\cdot\vec{p}+m]\,b_s(\vec{p})u_\lambda^s(\vec{p})}_{\gamma^0 p_0 u_\lambda^s(\vec{p})\,\to\,2\omega_p\delta^{ts}}\big] + \big[d_t(\vec{p})\underbrace{\bar{v}_\lambda^t(p)[\vec{\gamma}\cdot\vec{p}+m]\,d_s^\dagger(\vec{p})v_\lambda^s(\vec{p})}_{-\gamma^0 p_0 v_\lambda^s(\vec{p})\,\to\,2\omega_p\delta^{st}}\big]\Big\}$$

$$= \sum_{t,s}\frac{1}{\cancel{2\omega_p}}\int d\vec{p}\,\Big\{\big[b_t^\dagger(\vec{p})\,\omega_p\,\cancel{2\omega_p}\,\delta^{ts}\,b_s(\vec{p})\big] + \big[d_t(\vec{p})\,\omega_p\,\cancel{2\omega_p}\,\delta^{st}\,d_s^\dagger(\vec{p})\big]\Big\}$$

$$= \sum_s\int d\hat{p}\,\omega_p\,\big[b_s^\dagger(\vec{p})\,b_s(\vec{p}) + d_s(\vec{p})\,d_s^\dagger(\vec{p})\big] \underset{2^\circ\text{ paso}}{\rightsquigarrow} = \sum_s\int d\hat{p}\,\omega_p\,\big[b_s^\dagger(\vec{p})\,b_s(\vec{p}) + d_s^\dagger(\vec{p})\,d_s(\vec{p})\big]$$